

REPRESENTATION OF WEAKLY ADDITIVE OPERATORS¹

R. A. DECARLO AND R. SAEKS

ABSTRACT. This paper characterizes a class of nonlinear operators termed "weakly additive". A distributional kernel representation is constructed. A counterexample to a conjecture by Gersho is then given via the distributional kernel formulation.

I. Introduction. An operator, W , on a function space is weakly additive [1], [2] if $W(u + v) = W(u) + W(v)$ whenever u and v have disjoint supports. No homogeneity condition or additivity condition for functions with overlapping supports is required. This class of operators was first introduced by Zadeh [2] and has since been studied by Gersho [1], DeSantis [4], DeCarlo [5], and Saeks and Winslow [3] in the context of several applications to mathematical system theory.

Linear operators are weakly additive, as are unbiased (i.e. $M(0) = 0$) memoryless operators and operators obtained by composing a linear operator with an unbiased memoryless operator ($W = LM$). The weakly additive operators form a linear space which is complete in an appropriate sense [4]. Gersho [6] has conjectured that the space of weakly additive operators is generated by operators formed by composing a linear operator with a (possibly nonlinear) unbiased memoryless operator.

Given appropriate smoothness conditions, Gersho showed that a causal, continuous, translation invariant, weakly additive operator admits the "convolution-like" representation [1]:

$$(1.1) \quad y(t) = (Wx)(t) = \int_{-\infty}^{\infty} k(t - \tau, x(\tau)) d\tau$$

where $y(\cdot)$ and $x(\cdot)$ are elements of an appropriate function space and $k(\tau, s)$ is the derivative with respect to τ of $(Wsh)(\cdot)$ where $h(\cdot)$ is a Heaviside step.

The purpose of this paper is to exhibit a counterexample to Gersho's conjecture. In fact, upon recasting (1.1) in the setting of the Schwartz distributions, $k(t, s) = s\delta(t - s^2)$ becomes the required counterexample. Thus, in the process we extend (1.1) to arbitrary causal, translation invariant, continuous, weakly additive operators by allowing $k(t, s)$ to be distributional. The presence of the nonlinearity however, precludes the use of the traditional direct product formulation of the convolution. As such, we formulate our

Received by the editors June 20, 1975.

AMS (MOS) subject classifications (1970). Primary 46F10; Secondary 47H99 and 45H05.

Key words and phrases. Distribution, causal continuous translation invariant nonlinear operator, kernel, memoryless nonlinearity.

¹This research was supported in part by Air Force Office of Scientific Research Grant AFOSR-74-2631.

© American Mathematical Society 1976

nonlinear "convolution" as a weak* integral of an appropriate distribution valued function. The formulation is reminiscent of the classical function space derivation of the convolution integral.

In the following section the weak* integral is formulated and the various spaces of functions and distributions are introduced. The third section contains the derivation of our representation theorem and the fourth constructs the aforementioned counterexample.

II. Some topological vector spaces and the representation theorem. Our development is predicated upon the properties of several topological vector spaces.

(i) D is the usual space of testing functions, i.e. C^∞ functions with compact support with the inductive limit topology.

(ii) D' is the space of Schwartz distributions with the weak* topology.

(iii) L_1 is the space of integrable functions with the usual L_1 norm.

(iv) U is the space of bounded, continuous a.e. functions which have support (essentially) bounded on the left and a limit well defined a.e. on the right.

Finally, we will require a space of distribution valued functions of a real variable

$$(2.1) \quad f: R \rightarrow D'.$$

Such a function may be denoted variously by f or $\langle f, \cdot \rangle$. Its action on a testing function, ϕ , is denoted by $\langle f, \phi \rangle$. Note that $f(s)$ denotes the value of f at s , a distribution in D' . $\langle f(s), \phi \rangle$ denotes the real number resulting from the operation of this distribution on ϕ . To minimize confusion the underlying parameter of the testing function, ϕ , and the distribution, $f(s)$, will always be suppressed.

We say that f is continuous if it is continuous as a mapping from R to D' relative to the usual topologies of these spaces. Since D' has the weak* topology this is equivalent to requiring that the real valued function of a real variable $\langle f(s), \phi \rangle$ be continuous for each ϕ in D . Similarly, we say that f is integrable if $\langle f(s), \phi \rangle$ is integrable for every ϕ in D . f has (essentially) compact support if there exists an interval $[a, b]$ such that $f(s) = 0$ a.e. for s not in $[a, b]$ or equivalently that the function $\langle f(s), \phi \rangle$ have a common (essential) compact support independent of ϕ .

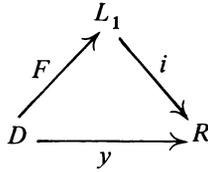
Finally the weak* integral of a distribution valued function, f , is a distribution y such that

$$(2.2) \quad \langle y, \phi \rangle = \int_{-\infty}^{\infty} \langle f(s), \phi \rangle ds$$

for all ϕ in D .

PROPOSITION. Define $F(\phi)(s) = \langle f(s), \phi \rangle$. A distribution valued function $f: R \rightarrow D'$ is weak* integrable if F maps D continuously into L_1 relative to the usual topologies of these spaces.

PROOF. The existence of the integral is equivalent to the requirement that the diagram



be well defined and commutative. Here i denotes the integral operator mapping L_1 onto R . Now if F maps D continuously into L_1 then the continuity of i assures that y is a well-defined distribution.

Another sufficient condition for weak* integrability was given by Zemanian [7].

PROPOSITION I. *Let f be Riemann integrable and have (essentially) compact support. Then f admits a weak* integral.*

By invoking the weak* integration concept discussed above we now formulate our generalization of equation (1.1) by representing an operator, W : $U \rightarrow D'$ as

$$(2.3) \quad W(u) = \int_{-\infty}^{\infty} \sigma_q k(u(q)) dq$$

where k is a distribution valued function. Here σ_s denotes the translation operator for a distribution defined by

$$(2.4) \quad \langle \sigma_s x, \phi \rangle = \langle x, \sigma_{-s} \phi \rangle.$$

For W to be well defined we must show that $\sigma_q k(u(q))$ is weak* integrable for all u in U . Our main theorem is as follows.

THEOREM. *Let K be a distribution valued function such that*

- (i) k is continuous (as a mapping from R to D'),
- (ii) $k(0) = 0$,
- (iii) $k(s)$ has support on the semi-infinite interval $[0, \infty)$ for all real s (i.e. if $\phi(r) = 0$ for $r > 0$ then $\langle k(s), \phi \rangle = 0$). Then

$$y = W(u) = \int_{-\infty}^{\infty} \sigma_q k(u(q)) dq$$

is a well-defined, continuous, causal, translation-invariant, weakly additive operator mapping U into D' . Conversely, any continuous, causal, translation-invariant, weakly additive operator may be so represented. Moreover, for any such operator $k(s) = \dot{W}(sh)$ (i.e. $k(s)$ is the distributional derivative of the response of W to a Heaviside step function, h , of height s).

By causality we mean that $(Wu_1)(t) = (Wu_2)(t)$ for almost all $t \leq T$ whenever $u_1(t) = u_2(t)$ for almost all $t \leq T$.

III. A sketch of the proof. The initial step is to show W is well defined given that $k(\cdot)$ satisfies the hypothesis of the theorem and that $u(\cdot)$ is in U . Equivalently show $\sigma_q k(u(q))$ is weak* integrable.

Under hypotheses (ii) and (iii), the fact that $u(q) = 0$ a.e. for $q \leq T_u$, and ϕ having compact support, the integrand, $\sigma_q k(u(q))$, will have compact support, $[T_u, b_\phi]$, where the support of ϕ is $[a_\phi, b_\phi]$. Since $u(\cdot)$ is uniformly bounded

and continuous a.e., and since $k(\cdot)$ is continuous, $\langle \sigma_q k(u(q)), \phi \rangle$ is continuous and integrable over $[T_u, b_\phi]$ for each ϕ in D -i.e.

$$(3.1) \quad \langle y, \phi \rangle = \int_{-\infty}^{\infty} \langle \sigma_q k(u(q)), \phi \rangle dq = \int_{T_u}^{b_\phi} \langle \sigma_q k(u(q)), \phi \rangle dq$$

exists for each ϕ in D . For (3.1) to define a distribution it must be continuous in ϕ . To this end consider the family of integrals of the form

$$(3.2) \quad y^B = \int_{T_u}^B \sigma_q k(u(q)) dq$$

where B is now independent of ϕ . Each of these objects exists and defines a distribution by Proposition I. Thus

$$(3.3) \quad \langle y, \phi \rangle = \lim_{B \rightarrow \infty} \langle y^B, \phi \rangle,$$

and by the completeness of D' , y is a distribution. The operator W is continuous because if a sequence $\{u_i\}$ of elements of U converges uniformly to u (over their common support (essentially) bounded on the left) then the continuity and integrability of $\sigma_q k(u_i(q))$ guarantees that the corresponding sequence of distributions, $\{y_i\}$, converges to the appropriate distribution y in the topology of D' .

The causality, translation-invariance, and weak additivity can be shown by a straightforward (though perhaps intricate) application of the hypotheses to the respective definitions.

The proof of the converse depends on the following Lemma. As such, a simple function is defined to be a finite linear combination of the characteristic functions of a sequence of nonoverlapping intervals; (t_0, t_1) , (t_1, t_2) , (t_2, t_3) , \dots , (t_{n-1}, t_n) where $t_0 > -\infty$ and $t_n = \infty$, i.e. m is simple if

$$(3.4) \quad m(q) = \sum_{i=1}^n a_i \chi_{[t_{i-1}, t_i)}(q)$$

where

$$(3.5) \quad -\infty < t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = \infty.$$

Equivalently, a simple function may be represented as a linear combination of shifted Heaviside step functions

$$(3.6) \quad m(q) = \sum_{i=1}^n b_i h(q - t_{i-1}) = \sum_{i=1}^n b_i \sigma_{t_{i-1}} h(q).$$

LEMMA. *The simple functions are dense in U .*

PROOF. Since u has a limit on the right well defined a.e., and since u is integrable, it can be approximated within ϵ almost everywhere on the interval $[T_u, T_\epsilon]$ by a simple function whose support lies in the interval [8] (i.e. a "classical" simple function wherein $t_0 = t_n$, $t_{n-1} = T_\epsilon$ and $a_n = 0$), say m_ϵ where

$$(3.7) \quad m_\epsilon(q) = \sum_{i=1}^{n-1} a_i \chi_{[t_{i-1}, t_i)}(q)$$

and

$$(3.8) \quad |m_\varepsilon(q) - u(q)| < \varepsilon \quad \text{a.e. } q \text{ in } [T_u, T_\varepsilon].$$

Thus a new simple function \mathbf{m} is defined by

$$(3.9) \quad \mathbf{m}_\varepsilon(q) = m_\varepsilon(q) + R_u \chi_{[T_u, \infty]}$$

which approximates u within ε almost everywhere on $[T_u, \infty]$. “ u ” and the simple functions so constructed all have a common support (essentially) bounded on the left by T_u . The \mathbf{m}_ε converge uniformly a.e. to u on this support. This verifies that the simple functions are dense in U .

Now let W be a continuous, causal, translation-invariant, weakly additive operator mapping U into D' . Define a distribution valued function, r , via $r(s) = W(sh)$ where h is the Heaviside step function, r is well defined since “ sh ” is in U for all s . Define a distribution valued function, k , by $k(s) = \dot{r}(s)$ where

$$(3.10) \quad \langle k(s), \phi \rangle = \langle \dot{r}(s), \phi \rangle = -\langle r(s), \dot{\phi} \rangle$$

(i.e. the distributional derivative). The causality, continuity, weak additivity and translation-invariance guarantee that “ r ” satisfies the hypotheses of the theorem and via equation (3.10) so does k . “ k ” thus defines a continuous, causal, translation-invariant, weakly additive operator, V , mapping U into D' via

$$(3.11) \quad V(u) = \int_{-\infty}^{\infty} \sigma_q k u(q) dq.$$

If $V(u) = W(u)$ the representation exists for the given operator, W . For this purpose it suffices to show that the two operators coincide on simple functions. Their continuity as mappings from U into D' then assures that they coincide on the entire space. In this task one shows that $V(sh) = W(sh) = r(s)$ by showing $\langle V(sh), \phi \rangle = \langle r(s), \phi \rangle$ for all ϕ in D . Now since both V and W are translation invariant and their step responses coincide so do their responses to the shifted step functions $\sigma_t sh$. Furthermore V and W coincide on the pulses $s\chi_{(a,b)}$ defined by

$$s\chi_{(a,b)}(q) = \begin{cases} s, & q \text{ in } (a, b), \\ 0, & q \text{ not in } (a, b). \end{cases}$$

This follows from the weak additivity of both W and V and the observation that

$$(3.13) \quad s\chi_{(a,b)} = \sigma_a sh - \sigma_b sh$$

since

$$(3.14) \quad \sigma_a sh = s\chi_{(a,b)} + \sigma_b sh$$

where the functions on the right side of (3.14) have disjoint support.

Finally, given a simple function

$$(3.15) \quad m = \sum_{i=1}^n a_i \chi_{[t_{i-1}, t_i]}$$

we note that the various terms making up “ m ” have nonoverlapping support and each such term is either a pulse or a step (the last term is a step since $t_n = \infty$). Thus the weak additivity of W implies that

$$(3.16) \quad W(m) = \sum_{i=1}^n W(a_i \chi_{[t_{i-1}, t_i)})$$

and the weak additivity of V implies that

$$(3.17) \quad V(m) = \sum_{i=1}^n V(a_i \chi_{[t_{i-1}, t_i)}).$$

The coincidences of the responses of V and W to pulse and steps implies that the terms on the right side of (3.16) coincide precisely with the terms on the right side of (3.17). As such,

$$(3.18) \quad W(m) = V(m)$$

for every simple function, m . Since the simple functions are dense in U and both W and V are continuous they must coincide on all of U since they coincide on a dense subset.

Finally, note that in the process of proving the converse it was shown that $k(s) = \dot{W}(sh)$.

IV. A counterexample. The purpose of this section is to give a counterexample to Gersho's conjecture to the effect that every weakly additive operator is a (possibly infinite) linear combination of operators of the form LM where L is linear and M is memoryless and unbiased, i.e.,

$$(4.1) \quad W = \sum_i L_i M_i$$

where the (possibly) infinite sum is defined with respect to the topology of the space of operators mapping U into D' . Since every linear operator is weakly additive our representation theorem applies to L_i with $k(s) = k_i s$ where k_i is a distribution in D' . This then yields the representation for L_i as

$$(4.2) \quad L_i(v) = \int_{-\infty}^{\infty} \sigma_q k_i v(q) dq$$

which coincides with the classical representation for a linear operator. Upon letting v be the response of a memoryless unbiased operator to u , i.e.,

$$(4.3) \quad v(q) = f_i(u(q))$$

where f_i is a continuous real valued mapping of a real variable such that $f_i(0) = 0$, we have

$$(4.4) \quad L_i M_i(u) = \int_{-\infty}^{\infty} \sigma_q k_i f_i(u(q)) dq.$$

Using this formula we may compute the step response of $L_i M_i$ via

$$(4.5) \quad \begin{aligned} \langle L_i M_i(sh), \phi \rangle &= \int_{-\infty}^{\infty} \langle \sigma_q k_i f_i(sh(q)), \phi \rangle dq = \int_0^{\infty} \langle k_i f_k(s), \sigma_{-q} \phi \rangle dq \\ &= \int_0^{\infty} f_i(s) \langle k_i, \sigma_{-q} \phi \rangle dq = - \int_0^{\infty} f_i(s) \langle r_i, \sigma_{-q} \phi \rangle dq \\ &= f_i(s) \left\langle f_i, \left[- \int_0^{\infty} \sigma_{-q} \phi dq \right] \right\rangle = f_i(s) \langle r_i, \phi \rangle. \end{aligned}$$

Here r_i is the primitive of k_i and the interchange of the integral and the

functional is justified by the fact that the function is linear and continuous and the fact that

$$(4.6) \quad - \int_0^{\infty} \sigma_{-q} \phi \, dq = \phi$$

is in D . As such, the step response of a linear combination of operators of the form $L_i M_i$ is of the form

$$(4.7) \quad \left\langle \left(\sum_i L_i M_i \right) (sh), \phi \right\rangle = \sum_i f_i(s) \langle r_i, \phi \rangle.$$

Now, consider the distribution valued function $k(s) = \sigma_{s^2} \dot{\delta}$ where $\dot{\delta}$ denotes the distributional derivative of the delta function. For any ϕ in D ,

$$(4.8) \quad \langle k(s), \phi \rangle = \langle \sigma_{s^2} \dot{\delta}, \phi \rangle = s \langle \dot{\delta}, \sigma_{-s^2} \phi \rangle = -s \langle \delta, \sigma_{-s^2} \dot{\phi} \rangle = -s \dot{\phi}(s^2)$$

is continuous in s showing that $k(s)$ is continuous. Clearly $k(0) = 0$ whereas the support of $k(s)$ is concentrated at s^2 which is always on the half-line. As such, k satisfies the hypotheses of the theorem and thus defines a continuous, causal, time-invariant, weakly additive operator W , whose step response is the primitive of $k(s)$, i.e. $W(sh) = \sigma_{s^2} \delta$ or equivalently

$$(4.9) \quad \langle W(sh), \phi \rangle = \langle \sigma_{s^2} \delta, \phi \rangle = s \phi(s^2).$$

Finally, a comparison of (4.7) and (4.9) will reveal that

$$(4.10) \quad \langle W(sh), \phi \rangle = s \phi(s^2) \neq \sum_i f_i(s) \langle r_i, \phi \rangle$$

for any choice of f_i and r_i since the "evaluation" of ϕ on the right side of (4.10) is independent of s whereas the evaluation of ϕ on the left side of (4.10) is dependent on s . W is therefore not of the required form and thus constitutes our counterexample.

Thus we have shown a counterexample to Gersho's conjecture. In the process we developed a representation theorem which clearly indicates that the class of weakly additive operators has a behavior very similar to linear operators.

REFERENCES

1. A. Gersho, *Nonlinear systems with a restricted additivity property*, IEEE Trans. Circuit Theory **16** (1969), 150-154.
2. L. A. Zadeh, *A contribution to the theory of nonlinear systems*, J. Franklin Inst. **255** (1953), 387-408. MR **15**, 376.
3. R. Saeks and L. Winslow, *Lossless nonlinear networks*, IEEE Trans. Circuit Theory **19** (1972), 392.
4. R. M. DeSantis, *Causality structure of engineering systems*, Ph. D. Thesis, Univ. of Michigan, Ann Arbor, Mich., 1971.
5. R. DeCarlo, *A distributional characterization of a weakly additive system*, M. S. Thesis, Univ. of Notre Dame, Notre Dame, Ind., 1973.
6. A. Gersho, Private communication.
7. A. H. Zemanian, *Distribution theory and transform analysis*, McGraw-Hill, New York, 1965. MR **31** # 1556.
8. M. Spivak, *Calculus on manifolds*, Benjamin, New York, 1965. MR **35** # 309.
9. R. Median, *Generalized impulse representation for time-varying networks*, IEEE Trans. Circuit Theory **19** (1972), 106-107.