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PSEUDOCOMPACTNESS PROPERTIES

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ABSTRACT. A topological extension property is a class of Tychonoff spaces $\mathcal{P}$ which is closed hereditary, closed under formation of topological products and contains all compact spaces. If X is Tychonoff and $\mathcal{P}$ is an extension property, there is a space $\mathcal{P}X$ such that $X \subseteq \mathcal{P}X \subseteq \beta X$, $\mathcal{P}X \in \mathcal{P}$ and if $f: X \to Y$ where $Y \in \mathcal{P}$ then f admits a continuous extension to $\mathcal{P}X$. A space X is called $\mathcal{P}$-pseudocompact if $\mathcal{P}X = \beta X$. In this note it is shown that if $\mathcal{P}$ is an extension property which contains the real line (e.g., the class of realcompact spaces), X is $\mathcal{P}$-pseudocompact and Y is compact, then $X \times Y$ is $\mathcal{P}$-pseudocompact. An example is given of an extension property $\mathcal{P}$, a $\mathcal{P}$-pseudocompact space X and a compact space Y such that $X \times Y$ is not $\mathcal{P}$-pseudocompact.

1. **Introduction.** If $\mathcal{P}$ is a class of completely regular, Hausdorff spaces which contains all compact spaces such that $\mathcal{P}$ is closed under the formation of topological products and closed subspaces, then $\mathcal{P}$ is called a topological extension property. Such classes may also be familiar as the almost-fitting epireflective subcategories of the category of completely regular, Hausdorff spaces. If X is a completely regular space and $\mathcal{P}$ is an extension property then there is a space $\mathcal{P}X$ such that $X \subseteq \mathcal{P}X \subseteq \beta X$, $\mathcal{P}X \in \mathcal{P}$, and every continuous map from X to a space in $\mathcal{P}$ admits a continuous extension to $\mathcal{P}X$. The basic references for this material are [5] and [9]. In what follows, a space will mean a completely regular, Hausdorff space. Given an extension property $\mathcal{P}$ and a space X, we call X $\mathcal{P}$-pseudocompact if $\mathcal{P}X = \beta X$. The class of $\mathcal{P}$-pseudocompact spaces is denoted by $\mathcal{P}'$. Woods [9] introduced this notion of $\mathcal{P}$-pseudocompactness and investigated the class $\mathcal{P}'$. If $\mathcal{P}$ is the class of realcompact spaces then $\mathcal{P}'$ is the class of pseudocompact spaces. In Theorem 2.2 of [9], it is shown that the class $\mathcal{P}'$, for an arbitrary extension property $\mathcal{P}$, satisfies many of the properties enjoyed by the class of pseudocompact spaces. It is well known that the topological product of a pseudocompact space with a compact space is pseudocompact (see 9.14 of [4]). It is also known (see Proposition 2.5 of [8]) that a space X is pseudocompact if and only if $E(X)$, the projective cover of X, is pseudocompact. In [9], the question is raised of whether or not the class $\mathcal{P}'$ satisfies the above two conditions for an arbitrary

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extension property $\mathcal{P}$. In this paper we show that the answer to both questions is, in general, no; however for a large class of extension properties $\mathcal{P}$ the answer to the first question is yes.

The notation will follows that of [4], and N will denote the natural numbers.

2. Pseudocompactness properties.

2.1. EXAMPLE. Let $\mathcal{P}$ be the class of spaces all of whose connected components are compact. Clearly $\mathcal{P}$ is an extension property. It is also easy to see that $\mathbf{R} \in \mathcal{P}'$ (where $\mathbf{R}$ is the real line). But $E(\mathbf{R})$ is extremally disconnected and noncompact. In particular, $E(\mathbf{R}) \in \mathcal{P}$ as $E(\mathbf{R})$ is totally disconnected, hence connected components of $E(\mathbf{R})$ are points. But by Theorem 2.2 of [9], we must have that $E(\mathbf{R}) \notin \mathcal{P}'$ as $E(\mathbf{R})$ is noncompact. This answers the second question mentioned in the introduction.

2.2. EXAMPLE. In what follows, let $\mathcal{P}$ denote the class of all spaces X such that every countable subset of X has compact closure in X. Such spaces are called $\aleph_0$-bounded spaces, and it is shown in [7] that $\mathcal{P}$ is an extension property and furthermore that if X is a space then $\mathcal{P}X$ consists of all points in βX which are in the closure of a countable subset of X. Clearly any separable space is in $\mathcal{P}'$, hence $N \in \mathcal{P}'$. We show that $N \times \beta D \notin \mathcal{P}'$ where D denotes the discrete space of cardinality $\aleph_1$.

Let $\{0, 1\}$ denote the two-point discrete space. Let X be the Σ-space of $\{0, 1\}^{\aleph_1}$, i.e., X consists of those points in $\{0, 1\}^{\aleph_1}$ which have at most countably many coordinates equal to 1. Then $X \in \mathcal{P}$. For each $i \in N$ define

$$A_i = \{p \in \{0, 1\}^{\aleph_1} : |\{k : \pi_k(p) = 1\}| \leqslant i\}$$

where π_k is the kth projection map from $\{0, 1\}^{\aleph_1}$ to $\{0, 1\}$. Clearly each A_i is compact, and if $A = \bigcup_{i \in N} A_i \subseteq X$ then A is dense in X. Thus $A \notin \mathcal{P}'$. For by Theorem 2.2 of [9], if $A \in \mathcal{P}'$ then $X \in \mathcal{P}'$, hence $X \in \mathcal{P} \cap \mathcal{P}'$ and X must be compact, which is not true. Since $|A_i| = \aleph_1$ for every $i \in N$, we have that A_i is the continuous image of βD. Thus $A = \bigcup_{i \in N} A_i$ is the continuous image of $N \times \beta D$. Since $A \notin \mathcal{P}'$ we have that $N \times \beta D \notin \mathcal{P}'$ by Theorem 2.2 of [9]. This answers question one.

2.3. THEOREM. *Suppose that $\mathcal{P}$ is an extension property such that $\mathbf{R} \in \mathcal{P}$, i.e., $\mathcal{P}$ contains all realcompact spaces. Then if $X \in \mathcal{P}'$ and Y is compact then $X \times Y \in \mathcal{P}'$.*

PROOF. By 2.2(e) of [9], every member of $\mathcal{P}'$ is pseudocompact. Let $X \in \mathcal{P}'$ and let Y be compact. Then $X \times Y$ is pseudocompact. Hence, by Theorem 2.1 of [3], $\beta(X \times Y) = \beta X \times \beta Y = \beta X \times Y$. Then by Proposition 3.1 of [2],

$$\mathcal{P}(X \times Y) = \mathcal{P}X \times \mathcal{P}Y = \mathcal{P}X \times Y = \beta X \times Y \quad (\text{as } X \in \mathcal{P}') = \beta(X \times Y),$$

i.e., $X \times Y \in \mathcal{P}'$. The condition that $\mathbf{R} \in \mathcal{P}$ is not necessary for the above result. If $\mathcal{P}$ is the class of compact spaces then $\mathbf{R} \notin \mathcal{P}$, but since $\mathcal{P}'$ consists of

all Tychonoff spaces, $\mathcal{P}'$ satisfies the condition that if $X \in \mathcal{P}'$ and Y is compact, then $X \times Y \in \mathcal{P}'$. $\square$

3. **0-dimensional pseudocompactness properties.** If X is 0-dimensional (i.e., has a base of clopen sets), let $\beta_0 X$ denote the Banaschewski maximal 0-dimensional compactification of X (see [1]). The space $\beta_0 X$ is the Stone space (maximal ideal space) of the Boolean algebra of clopen subsets of X. If $\mathcal{P}_0$ is a class of 0-dimensional spaces which contains $\{0, 1\}$, is closed hereditary and preserved under the formation of products (an epireflective, almost fitting subcategory of the category of 0-dimensional spaces), then $\mathcal{P}_0$ is called a 0-dimensional extension property. If X is 0-dimensional and $\mathcal{P}_0$ is a 0-dimensional extension property, then there is a space $\mathcal{P}_0 X$ such that $X \subseteq \mathcal{P}_0 X \subseteq \beta_0 X$, $\mathcal{P}_0 X \in \mathcal{P}_0$, and any continuous map from X to a space in $\mathcal{P}_0$ admits a continuous extension to $\mathcal{P}_0 X$. This space can be produced in a way analogous to the construction of $\mathcal{P}X$ in [5]. The concept of $\mathcal{P}_0$-pseudocompactness is analogous to that of $\mathcal{P}$-pseudocompactness defined above. Since $N \times \beta D$ is 0-dimensional, in view of Example 2.2 above, question one asked in the introduction has a negative answer when applied to 0-dimensional extension properties. However, Example 2.1 above will not work in the 0-dimensional case of question two. The following example shows that the answer to question two with respect to 0-dimensional extension properties is negative.

3.1. EXAMPLE. Let $\omega_1^* = \omega_1 \cup \{\omega_1\}$ denote the one-point compactification of ω_1, the totally ordered space of countable ordinals. Let $\mathcal{P}_0$ be the class of all 0-dimensional, $\aleph_0$-bounded spaces. Then $X = N \times \omega_1^* \in \mathcal{P}_0'$ (if $p \in \beta_0 X$ and p is not in the closure of $N \times \{\omega_1\}$, then p is in the closure of a complement of a neighborhood of $N \times \{\omega_1\}$ which must be countable). For each $i \in N$ let f_i be a bijection between D, the discrete space of cardinality $\aleph_1$, and the isolated points of ω_1^*, and let $f_i^*: \beta D \to \omega_1^*$ be the Stone extension of f_i. Then f_i^* is perfect and irreducible (i.e., no proper closed subset of βD maps onto ω_1^*). Thus $\bigcup_{i \in N} f_i: N \times \beta D \to N \times \omega_1^*$ is perfect and irreducible. By Lemma 2.14 of [8], $N \times \beta D = E(N \times \omega_1^*)$. However, Example 2.2 above shows that $N \times \beta D \notin \mathcal{P}_0'$ (we showed $N \times \beta D \notin \mathcal{P}'$ where $\mathcal{P}$ is the class of $\aleph_0$-bounded spaces, but since $N \times \beta D$ is extremally disconnected we have $\beta_0(N \times \beta D) = \beta(N \times \beta D)$ and, hence, $\mathcal{P}_0(N \times \beta D) = \mathcal{P}(N \times \beta D) \subsetneq \beta(N \times \beta D)$ by Example 2.2).

We can sharpen the result of Theorem 2.3 when restricting ourselves to 0-dimensional extension properties.

3.2 THEOREM. *Let $\mathcal{P}_0$ be a 0-dimensional extension property not all of whose members are pseudocompact. Let $X \in \mathcal{P}_0'$ and let Y be compact and 0-dimensional. Then $X \times Y \in \mathcal{P}_0'$.*

PROOF. Since not every member of $\mathcal{P}_0$ is pseudocompact it follows that some members of $\mathcal{P}_0$ contain a C-embedded, hence closed, copy of N (see 1.21 and 3B3 of [4]). Thus $N \in \mathcal{P}_0$. By Theorem 2.2 of [9], every member of $\mathcal{P}_0'$ is pseudocompact.

Let $X \in \mathcal{P}'_0$ and Y be compact and 0-dimensional. Then $X \times Y$ is pseudocompact. Hence by Theorem 2.2 of [2],

$$\beta_0(X \times Y) = \beta_0 X \times \beta_0 Y = \beta_0 X \times Y.$$

Then by Proposition 3.1 of [2],

$$\mathcal{P}_0(X \times Y) = \mathcal{P}_0 X \times \mathcal{P}_0 Y = \mathcal{P}_0 X \times Y = \beta_0 X \times Y = \beta_0(X \times Y),$$

hence $X \times Y \in \mathcal{P}'_0$. $\square$

Corollary 2.10 of [9] shows that if $\mathcal{P}_0$ is a 0-dimensional extension property, then either $\mathcal{P}_0$ is the class of compact 0-dimensional spaces or $\mathcal{P}'_0$ does not properly contain the class of pseudocompact 0-dimensional spaces. The question is then raised of whether or not the corresponding statement for (not necessarily 0-dimensional) extension properties holds. If $\mathcal{P}$ is the class of $I \times N$-compact spaces (if E is a space, then X is E-compact if X can be embedded as a closed subset of E^m for some cardinal number m; see [6]), then $\mathcal{P}$ is an extension property. But clearly $\mathcal{P}$ contains noncompact spaces (e.g., $N \in \mathcal{P}$). Also $\mathcal{P}'$ properly contains the class of pseudocompact spaces. This follows from 2.2(e) of [9] and the fact that $\mathbf{R} \in \mathcal{P}'$, and $\mathbf{R}$ is not pseudocompact. Thus the statement for arbitrary extension properties corresponding to 2.10 of [9] is not valid.

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