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DEFICIENCIES OF THE ASSOCIATED CURVES OF A HOLOMORPHIC CURVE IN THE PROJECTIVE SPACE

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ABSTRACT. Let ${}_k x$ be the nonconstant associated holomorphic curve of rank k $(1 \leqq k \leqq n-1)$ of a transcendental holomorphic curve $x\colon \mathbf{C} \to P_n\mathbf{C}$. It is proved that if $1 \leqq k \leqq n-2$ and $A_j^k \in P_{l(k)-1}\mathbf{C}, j = 1, \ldots, 2l(k) - 2$ $(l(k) = \binom{n+1}{k+1})$ are in general position and $\langle {}_k x, A_j^k \rangle \not\equiv 0$ for all A_j^k, then $\sum_{j=1}^{2l(k)-2} \delta_k(A_j^k) \leqq 2l(k) - 3$ and that in the case when $k = n-1$, $\sum_{A^{n-1}} \delta_{n-1}(A^{n-1}) \leqq l(n-1)$, where $\{A^{n-1}\}$ is a finite subset of $P_{l(n-1)-1}\mathbf{C}$ in general position such that $\langle {}_{n-1}x, A^{n-1} \rangle \not\equiv 0$ for all A^{n-1}. These are sharp.

1. The theory of holomorphic (or meromorphic) curves was initiated by H. and J. Weyl in 1938 [7], [8]. Its main problem, the proof of the so-called defect relations, was solved by Ahlfors [1]. A modern detailed treatment was given by Wu [9]. It is a natural generalization of the Picard-Borel-Nevanlinna theory. On the other hand, a generalization of the Nevanlinna theory to systems of entire functions had been tried by Cartan in 1933 [2].

If a system of entire functions $(x_0, x_1, \ldots, x_n)$ is given, then we can see it as a reduced representation of a holomorphic curve $x\colon \mathbf{C} \to P_n\mathbf{C}$. By the reasoning in [2, pp. 7–8] and [8, pp. 81–84] or [9, pp. 104–105], the characteristic function defined by Cartan for the system of entire functions is essentially equal to the order function defined by Weyl for the corresponding holomorphic curve in the projective space. Hence the results on systems of entire functions can be restated by the statements on holomorphic curves in the projective space.

The defect relation for nondegenerate holomorphic curves in the projective space was obtained by Ahlfors [1] and Cartan [2]. But it seems that a defect relation for degenerate holomorphic curves in the projective space has not been yet obtained in a sharp form. Concerning it there are a result of Toda [5] and a conjecture of Cartan [2].

Recently, Toda [6] made another contribution to the study of degenerate holomorphic curves which is in turn a generalization of the result of the author and Ozawa [3], [4] on entire algebroid functions. In the language of holomorphic curves (cf. [9]), Toda's result concerns the defects of the original curve.

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This paper makes a similar study for the defects of the associated curves.

A result of Toda [6] is restated by the following statement on holomorphic curves in the projective space.

THEOREM A. *Let* $x\colon \mathbf{C} \to P_n\mathbf{C}$ *be a transcendental holomorphic curve and* $\tilde{x} = (x_0, x_1, \ldots, x_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ *its reduced representation. If there are* $a_j \in P_n\mathbf{C}, j = 1, 2, \ldots, 2n$, *in general position such that* $\langle x, a_j\rangle \not\equiv 0$ *for all* j *and* $\sum_{j=1}^{2n} \delta_0(a_j) > 2n - 1$, *then* $\lambda = n - 1$, *where* λ *is the maximum number of independent, linear relations with constant coefficients among the entire functions* $x_0, x_1, \ldots, x_n$, *and the entire functions* $\langle x(z), a_j\rangle$ *are divided into two classes having the following properties*:

(i) *either class contains n entire functions*,

(ii) *the entire functions of the same class are proportional.*

(For terminology and notations of the theory of holomorphic curves, see Wu's note [9].)

As corollary of Theorem A, we have

THEOREM B. *Let* $x\colon \mathbf{C} \to P_n\mathbf{C}$ *be a transcendental holomorphic curve. If* $a_j \in P_n\mathbf{C}, j = 1, 2, \ldots, 2n + 1$, *are in general position and* $\langle x, a_j\rangle \not\equiv 0$ *for all* j, *then we have* $\sum_{j=1}^{2n+1} \delta_0(a_j) \leqq 2n$.

Theorem B is sharp in the sense that there are a holomorphic curve $x\colon \mathbf{C} \to P_n\mathbf{C}$ and $a_j \in P_n\mathbf{C}, j = 1, \ldots, 2n$, in general position such that $\langle x, a_j\rangle \not\equiv 0$ for all j and

$$\sum_{j=1}^{2n} \delta_0(a_j) = 2n. \tag{1.1}$$

In fact, put $x_0(z) = e^z - 1$, $x_1(z) \equiv \cdots \equiv x_{n-1}(z) \equiv 0$, $x_n(z) = -e^z + 2$ and $\tilde{a}_j = (\alpha_j^n, \alpha_j^{n-1}, \ldots, \alpha_j, 1)$, where $\eta = \exp(2\pi i/n)$ and $\alpha_j = \eta^{j-1}$ for $j = 1, 2, \ldots, n$ and $\alpha_j = 2^{1/n}\eta^{j-n-1}$ for $j = n + 1, \ldots, 2n$. Then $\tilde{x} = (x_0, x_1, \ldots, x_n)$ satisfies $\langle \tilde{x}(z), \tilde{a}_j\rangle = 1$ for $j = 1, \ldots, n$ and $\langle \tilde{x}(z), \tilde{a}_j\rangle = e^z$ for $j = n + 1, \ldots, 2n$. It is clear that $\pi(\tilde{a}_j) = a_j \in P_n\mathbf{C}, j = 1, \ldots, 2n$, are in general position and the holomorphic curve $x\colon \mathbf{C} \to P_n\mathbf{C}$ induced by $\tilde{x}$ satisfies $\langle x(z), a_j\rangle \neq 0$ for all $z \in \mathbf{C}$ and $j = 1, \ldots, 2n$. Hence we have $\delta_0(a_j) = 1$ for $j = 1, \ldots, 2n$ and so (1.1). Thus we obtain a desired example.

In this paper we shall prove

THEOREM 1. *Let* ${}_kx\colon \mathbf{C} \to G(n,k) \subseteqq P_{l(k)-1}\mathbf{C}$ $(l(k) = \binom{n+1}{k+1})$ *be the nonconstant associated holomorphic curve of rank* k $(1 \leqq k \leqq n - 2)$ *of a transcendental holomorphic curve* $x\colon \mathbf{C} \to P_n\mathbf{C}$. *If* $A_j^k \in P_{l(k)-1}\mathbf{C}, j = 1, 2, \ldots, 2l(k) - 2$, *are in general position and* $\langle {}_kx, A_j^k\rangle \not\equiv 0$ *for all* j, *then we have*

$$\sum_{j=1}^{2l(k)-2} \delta_k(A_j^k) \leqq 2l(k) - 3.$$

Theorem 1 is sharp in the following sense:

THEOREM 2. *There are a holomorphic curve* $x\colon \mathbf{C} \to P_n\mathbf{C}$ *and* $A_j^k \in P_{l(k)-1}\mathbf{C}$, $j = 1, 2, \ldots, 2l(k) - 3$, *in general position such that* $\langle {}_k x, A_j^k \rangle \not\equiv 0$ *for all* j *and*

$$\sum_{j=1}^{2l(k)-3} \delta_k(A_j^k) = 2l(k) - 3 \tag{1.2}$$

for some n *and* k.

Theorem 1 is not true in the case $k = 0$. Because $2l(k) - 2 = 2n$ for $k = 0$ and we have already had a holomorphic curve $x\colon \mathbf{C} \to P_n\mathbf{C}$ satisfying (1.1).

For the case $k = n - 1$, we shall prove

THEOREM 3. *Let* ${}_{n-1}x\colon \mathbf{C} \to G(n, n-1) \subsetneqq P_{l(n-1)-1}\mathbf{C}$ *be the nonconstant associated holomorphic curve of rank* $n - 1$ *of a transcendental holomorphic curve* $x\colon \mathbf{C} \to P_n\mathbf{C}$. *If* $\{A^{n-1}\}$ *is a finite subset of* $P_{l(n-1)-1}\mathbf{C}$ *in general position and* $\langle {}_{n-1}x, A^{n-1} \rangle \not\equiv 0$ *for all* A^{n-1}, *then*

$$\sum_{A^{n-1}} \delta_{n-1}(A^{n-1}) \leqq l(n-1).$$

Theorem 3 is sharp in the following sense:

THEOREM 4. *There are a holomorphic curve* $x\colon \mathbf{C} \to P_n\mathbf{C}$ *and* $A_j^{n-1} \in P_{l(n-1)-1}\mathbf{C}$, $j = 1, 2, \ldots, l(n-1)$, *in general position such that* $\langle {}_{n-1}x, A_j^{n-1} \rangle \not\equiv 0$ *for all* j *and*

$$\sum_{j=1}^{l(n-1)} \delta_{n-1}(A_j^{n-1}) = l(n-1) \tag{1.3}$$

for some n.

REMARKS. Chen-Han Sung[1] has recently proved the conjecture of Cartan [2], that is,

THEOREM C. *Let* $x\colon \mathbf{C} \to P_n\mathbf{C}$ *be a transcendental holomorphic curve,* $\tilde{x} = (x_0, x_1, \ldots, x_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ *its reduced representation and* λ *the maximum number of independent, linear relations with constant coefficients among entire functions* $x_0, x_1, \ldots, x_n$. *If* $\{a\}$ *is a finite subset of* $P_n\mathbf{C}$ *in general position and* $\langle x, a \rangle \not\equiv 0$ *for all* a, *then*

$$\sum_a \delta_0(a) \leqq n + 1 + \lambda.$$

We can deduce from Theorem C, the proof of our Theorem 1 given in §2 and our Theorem 2 that

THEOREM 1′. *Let* ${}_k x\colon \mathbf{C} \to G(n, k) \subsetneqq P_{l(k)-1}\mathbf{C}$ *be the nonconstant associated holomorphic curve of rank* k $(1 \leqq k \leqq n-2)$ *of a transcendental holomorphic curve* $x\colon \mathbf{C} \to P_n\mathbf{C}$. *If* $\{A^k\}$ *is a finite subset of* $P_{l(k)-1}\mathbf{C}$ *in general position and* $\langle {}_k x, A^k \rangle \not\equiv 0$ *for all* A^k, *then*

[1] Communication with the referee.

$$\sum_{A^k} \delta_k(A^k) \leqq 2l(k) - 3.$$

This is sharp.

The author wishes to express his heartiest thanks to the referee for his valuable advice and pointing out the result of Sung.

2. **Proof of Theorem 1.** Let $\tilde{x} = (x_0, x_1, \ldots, x_n)$ be a reduced representation of x. We denote by λ the maximum number of independent, linear relations with constant coefficients among $x_0, \ldots, x_n$. Then there are $n + 1 - \lambda$ linearly independent functions in $\{x_j\}$, say $x_0, x_1, \ldots, x_{n-\lambda}$, such that the other functions are written by their linear combinations. Hence every ordered component of $X^k = \tilde{x} \wedge \tilde{x}^{(1)} \wedge \cdots \wedge \tilde{x}^{(k)}$ is written by a linear combination of $\binom{n+1-\lambda}{k+1}$ ordered components

$$X^k(i_0\, i_1 \cdots i_k) = \|x_{i_0}\, x_{i_1} \cdots x_{i_k}\|, \qquad 0 \leqq i_0 < i_1 < \cdots < i_k \leqq n - \lambda,$$

of X^k, where $\|x_{i_0} \cdots x_{i_k}\|$ is the Wronskian of $x_{i_0}, \ldots, x_{i_k}$. Since ${}_k x\colon \mathbf{C} \to G(n,k) \subseteqq P_{l(k)-1}\mathbf{C}$ is not constant, we have $\binom{n+1-\lambda}{k+1} \geqq 2$ and, consequently,

$$0 \leqq \lambda \leqq n - k - 1. \tag{2.1}$$

Now suppose, to the contrary, that $\sum_{j=1}^{2l(k)-2} \delta_k(A_j^k) > 2l(k) - 3$. Then we apply Theorem A to the holomorphic curve ${}_k x\colon \mathbf{C} \to P_{l(k)-1}\mathbf{C}$. Theorem A implies $\lambda_k = l(k) - 2$, where λ_k is the maximum number of independent, linear relations with constant coefficients among $l(k)$ ordered components $\{X^k(i_0 \cdots i_k)\}$ of X^k. Hence there are two linearly independent components of X^k in $\{X^k(i_0 \cdots i_k)\}$, $0 \leqq i_0 < \cdots < i_k \leqq n - \lambda$, say $X^k(0 \cdots k)$ and $X^k(j_0 \cdots j_k)$, such that the other components are written by their linear combinations. (2.1) and $k \geqq 1$ imply $\binom{n+1-\lambda}{k+1} \geqq 3$.

Assume that $\binom{n+1-\lambda}{k+1} = 3$. Then we have $n + 1 - \lambda = k + 2 = 3$ and so $n - \lambda = 2$ and $k = 1$. Since every ordered component of X^1 is written by a linear combination of $X^1(0\ 1) = \|x_0\, x_1\|$ and $X^1(j_0 j_1) = \|x_{j_0}\, x_{j_1}\|$, $0 \leqq j_0 < j_1 \leqq n - \lambda = 2$, we have

$$a\|x_0\, x_1\| + b\|x_0\, x_2\| + c\|x_1\, x_2\| = 0$$

and so $\{\|bx_0 + cx_1\; ax_1 + bx_2\|\}/b = 0$ with three suitable constants a, b and c with $(a,b,c) \neq (0,0,0)$, say $b \neq 0$. Hence $bx_0 + cx_1$ and $ax_1 + bx_2$, that is, x_0, x_1 and x_2 are linearly dependent, which contradicts our assumption that x_0, x_1 and x_2 $(n - \lambda = 2)$ are linearly independent.

Next assume that $\binom{n+1-\lambda}{k+1} \geqq 4$. Then since $3 \leqq k + 1 \leqq n - \lambda$, we have

$$\|x_0 \cdots x_{k-2}\, x_{k-1}\, x_{k+1}\| = a\|x_0 \cdots x_k\| + b\|x_{j_0} \cdots x_{j_k}\|,$$

$$\|x_0 \cdots x_{k-2}\, x_k\, x_{k+1}\| = c\|x_0 \cdots x_k\| + d\|x_{j_0} \cdots x_{j_k}\|$$

with suitable constants a, b, c and d. If $ad - bc = 0$, then we have

$$\|x_0 \cdots x_{k-2}\, x_{k-1}\, x_{k+1}\| + \alpha\|x_0 \cdots x_{k-2}\, x_k\, x_{k+1}\| = 0$$

and so

$$\|x_0 \cdots x_{k-2}\, x_{k-1} + \alpha x_k\, x_{k+1}\| = 0$$

with a suitable constant α. Hence $x_0, \ldots, x_{k-2}, x_{k-1} + \alpha x_k$ and x_{k+1}, that is, $x_0, \ldots, x_{k+1}$ are linearly dependent, which contradicts our assumption that $x_0, \ldots, x_{n-\lambda}$ $(k + 1 \leqq n - \lambda)$ are linearly independent. If $ad - bc \neq 0$, then we have

$$\alpha\|x_0 \cdots x_k\| - d\|x_0 \cdots x_{k-2}\, x_{k-1}\, x_{k+1}\| - b\|x_0 \cdots x_{k-2}\, x_k\, x_{k+1}\| = 0$$

and so

$$\|x_0 \cdots x_{k-2}\, dx_{k-1} + bx_k\, \alpha x_k - dx_{k+1}\| = 0,$$

where $\alpha = ad - bc \neq 0$. Hence $x_0, \ldots, x_{k-2}, dx_{k-1} + bx_k$ and $\alpha x_k - dx_{k+1}$, that is, $x_0, \ldots, x_{k+1}$ are linearly dependent, which is a contradiction.

Therefore we obtain

$$\sum_{j=1}^{2l(k)-2} \delta_k(A_j^k) \leqq 2l(k) - 3,$$

which gives our Theorem 1.

3. **Proof of Theorem 2.** For $n = 3$ and $k = 1$ we shall give a holomorphic curve x: $\mathbf{C} \to P_n\mathbf{C}$ and $A_j^k \in P_{l(k)-1}\mathbf{C}, j = 1, \ldots, 2l(k) - 3$, in general position satisfying (1.2). We have $l(k) = 6$ and $2l(k) - 3 = 9$ for $n = 3$ and $k = 1$.

Put

$$x_0(z) = 15^{1/2} ize^{z/2}, \qquad x_1(z) = 15^{1/2} ie^{z/2},$$

$$x_2(z) = -(1/15)^{1/2} i(e^z - 1)e^{z/2}, \qquad x_3(z) = 16(1/15)^{1/2} i(e^z - 1)e^{z/2}.$$

Let x: $\mathbf{C} \to P_3\mathbf{C}$ be the holomorphic curve induced by $\tilde{x} = (x_0, x_1, x_2, x_3)$. Then $X^1 = \tilde{x} \wedge \tilde{x}^{(1)}$ has the following ordered components:

$$X^1(0\ 2) = \|x_0\, x_2\| = x_0 x_2' - x_0' x_2 = e^z + (z - 1)e^{2z}, \quad X^1(1\ 2) = e^{2z},$$

$$X^1(0\ 1) = 15e^z, \quad X^1(0\ 3) = -16(e^z + (z - 1)e^{2z}), \quad X^1(2\ 3) = 0,$$

$$X^1(1\ 3) = -16e^{2z}.$$

Considering X^1 as a point of $\mathbf{C}^6$, we put $X^1 = (X^1(0\ 2), X^1(1\ 2), X^1(0\ 1), X^1(2\ 3), X^1(0\ 3), X^1(1\ 3)) \in \mathbf{C}^6$. Further put $\tilde{A}_j^1 = (\overline{w}_j^5, \overline{w}_j^4, \ldots, \overline{w}_j, 1) \in \mathbf{C}^6$. Then we have

$$\begin{aligned}\langle X^1, \tilde{A}_j^1\rangle &= (e^z + (z-1)e^{2z})w_j^5 + e^{2z}w_j^4 + 15e^z w_j^3 \\ &\quad -16(e^z + (z-1)e^{2z})w_j - 16e^{2z} \\ &= w_j(w_j^2 - 1)(w_j^2 + 16)e^z + (w_j^4 - 16)((z-1)w_j + 1)e^{2z}.\end{aligned}$$

Put $w_1 = 0$, $w_2 = 1$, $w_3 = -1$, $w_4 = 4i$, $w_5 = -4i$, $w_6 = 2$, $w_7 = -2$, $w_8 = 2i$ and $w_9 = -2i$. Then it is clear that $A_j^1 = {}_1\pi(\tilde{A}_j^1) \in P_5\mathbf{C}$, $j = 1, \ldots, 9$, are in general position. And we have

$$\langle X^1, \tilde{A}_1^1\rangle = -16e^{2z}, \quad \langle X^1, \tilde{A}_2^1\rangle = -15ze^{2z}, \quad \langle X^1, \tilde{A}_3^1\rangle = 15(z-2)e^{2z},$$

$$\langle X^1, \tilde{A}_4^1\rangle = 240(4iz + 1 - 4i)e^{2z}, \quad \langle X^1, \tilde{A}_5^1\rangle = -240(4iz - 1 - 4i)e^{2z},$$

$$\langle X^1, \tilde{A}_6^1\rangle = 120e^z, \quad \langle X^1, \tilde{A}_7^1\rangle = -120e^z, \quad \langle X^1, \tilde{A}_8^1\rangle = -120ie^z,$$

$$\langle X^1, \tilde{A}_9^1\rangle = 120ie^z.$$

Thus $\langle X^1, \tilde{A}_j^1\rangle$ has no zero or only one zero. Hence $\delta_1(A_j^1) = 1$ for $j = 1, \ldots, 9$, which gives (1.2) for $n = 3$ and $k = 1$.

Thus the proof of Theorem 2 is complete.

4. **Proof of Theorem 3.** Suppose, to the contrary, that $\sum_{A^{n-1}} \delta_{n-1}(A^{n-1}) > l(n-1)$. Then we apply Theorem 5.13 (defect relations) in [9] to the holomorphic curve ${}_{n-1}x\colon \mathbf{C} \to P_{l(n-1)-1}\mathbf{C}$. It follows that the holomorphic curve ${}_{n-1}x$ must be degenerate, that is, there is a linear relation among the ordered components of $X^{n-1} = \tilde{x} \wedge \cdots \wedge \tilde{x}^{(n-1)}$, where $\tilde{x} = (x_0, \ldots, x_n)$ is a reduced representation of x. Hence we have

$$\sum_{j=0}^{n} \alpha_j X^{n-1}(0 \cdots \hat{j} \cdots n) = \sum_{j=0}^{n} \alpha_j \|x_0 \cdots \hat{x}_j \cdots x_n\| = 0, \tag{4.1}$$

where $\hat{}$ means "omit". We denote by $h + 1$ $(0 \leqq h \leqq n)$ the number of nonzero α_j. Then we may assume, without loss of generality, that $\alpha_0\alpha_1 \cdots \alpha_h \neq 0$. Then from (4.1) we have

$$\sum_{j=0}^{h} \alpha_j \|x_0 \cdots \hat{x}_j \cdots x_n\| = 0$$

and consequently

$$\frac{\|\alpha_1 x_0 + \alpha_0 x_1\ \alpha_2 x_1 + \alpha_1 x_2 \cdots \alpha_h x_{h-1} + \alpha_{h-1} x_h\ x_{h+1} \cdots x_n\|}{(\alpha_1 \cdots \alpha_{h-1})} = 0.$$

Therefore $\alpha_1 x_0 + \alpha_0 x_1, \ldots, \alpha_h x_{h-1} + \alpha_{h-1} x_h, x_{h+1}, \ldots, x_n$, that is, $x_0, \ldots, x_n$ are linearly dependent. Then we deduce that the ordered components of X^{n-1} are proportional. Hence ${}_{n-1}x\colon \mathbf{C} \to G(n, n-1) \subseteqq P_{l(n-1)-1}\mathbf{C}$ is a constant map, which contradicts our assumption. Therefore we have

$$\sum_{A^{n-1}} \delta_{n-1}(A^{n-1}) \leqq l(n-1),$$

which gives our Theorem 3.

5. **Proof of Theorem 4.** For $n = 3$ we shall give a holomorphic curve $x\colon \mathbf{C} \to P_n\mathbf{C}$ and $A_j^{n-1} \in P_{l(n-1)-1}\mathbf{C}, j = 1, \ldots, l(n-1)$, in general position satisfying $\langle {}_{n-1}x, A_j^{n-1}\rangle \not\equiv 0$ and (1.3). We have $l(n-1) = 4$ for $n = 3$.

Put

$$x_0(z) = (-3e^{2z} - 9e^z + 3 + e^{-z})/18^{2/3}, \qquad x_1(z) = (9e^z + 3 - 2e^{-z})/18^{2/3},$$

$$x_2(z) = (-9e^z + 3 + 4e^{-z})/18^{2/3}, \qquad x_3(z) = (9e^z + 3 - 8e^{-z})/18^{2/3}.$$

Let $x\colon \mathbf{C} \to P_3\mathbf{C}$ be the holomorphic curve induced by $\tilde{x} = (x_0, x_1, x_2, x_3)$. Then $X^2 = \tilde{x} \wedge \tilde{x}^{(1)} \wedge \tilde{x}^{(2)}$ has the following ordered components:

$$\begin{aligned} X^2(0\ 1\ 2) &= \|x_0\ x_1\ x_2\| \\ &= x_0x_1'x_2'' + x_1x_2'x_0'' + x_2x_0'x_1'' - x_0''x_1'x_2 - x_1''x_2'x_0 - x_2''x_0'x_1 \\ &= e^{3z} + e^{2z} + e^z + 1, \end{aligned}$$

$$X^2(0\ 1\ 3) = -3e^{2z} - e^z - 2, \quad X^2(0\ 2\ 3) = -e^{3z} + 2e^{2z} - 2e^z - 1,$$

$$X^2(1\ 2\ 3) = 2.$$

Considering X^2 as a point of $\mathbf{C}^4$, we put $X^2 = (X^2(0\ 1\ 2), X^2(0\ 1\ 3), X^2(0\ 2\ 3), X^2(1\ 2\ 3)) \in \mathbf{C}^4$. Further put $\tilde{A}_j^2 = (\overline{w}_j^3, \overline{w}_j^2, \overline{w}_j, 1) \in \mathbf{C}^4$. Then we have

$$\begin{aligned} \langle X^2, \tilde{A}_j^2\rangle &= (e^{3z} + e^{2z} + e^z + 1)w_j^3 + (-3e^{2z} - e^z - 2)w_j^2 \\ &\quad + (-e^{3z} + 2e^{2z} - 2e^z - 1)w_j + 2 \\ &= w_j(w_j - 1)(w_j + 1)e^{3z} + w_j(w_j - 1)(w_j - 2)e^{2z} \\ &\quad + w_j(w_j + 1)(w_j - 2)e^z + (w_j - 1)(w_j + 1)(w_j - 2). \end{aligned}$$

Put $w_1 = 0$, $w_2 = 1$, $w_3 = -1$ and $w_4 = 2$. Then it is clear that $A_j^2 = {}_2\pi(\tilde{A}_j^2) \in P_3\mathbf{C}, j = 1, \ldots, 4$, are in general position. And we have

$$\langle X^2, \tilde{A}_1^2\rangle = 2, \quad \langle X^2, \tilde{A}_2^2\rangle = -2e^z,$$

$$\langle X^2, \tilde{A}_3^2\rangle = -6e^{2z}, \quad \langle X^2, \tilde{A}_4^2\rangle = 6e^{3z}.$$

Thus $\langle X^2, \tilde{A}_j^2\rangle$ has no zero. Hence $\delta_2(A_j^2) = 1$ for $j = 1, \ldots, 4$, which gives (1.3) for $n = 3$. Thus the proof of Theorem 4 is complete.

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