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ABSOLUTE SCHAUDER BASES FOR $C(X)$ WITH THE COMPACT-OPEN TOPOLOGY

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ABSTRACT. Let X be a completely regular Hausdorff space and $C(X)$ the space of all real valued continuous functions on X. We give $C(X)$ the compact-open topology. The principal result of this paper is: If $C(X)$ is complete and barrelled and has an absolute basis, then $C(X)$ is isomorphic to a countable product of lines.

Let X be a completely regular Hausdorff space and let $C(X)$ be the space of all real valued and continuous functions on X. Throughout this note we give $C(X)$ the compact-open topology, unless otherwise indicated, and the notation is essentially that of Bourbaki; see, e.g., [2] or [12].

We call a set A in a locally convex space E *total* if $f \in E'$ and $f(A) = 0$ implies $f = 0$. Recall that the dual of $C(X)$ with the compact-open topology is the space of all signed regular Borel measures on X having compact support; see, e.g., [6]. We denote this space by $M(X)$ or M.

1. PROPOSITION. *When X is locally compact the following are equivalent*:

(1) *$C(X)$ is separable and M with $\sigma(M, C)$ topology is separable.*

(2) *X is a separable metric space.*

If (1) *or* (2) *holds, then $C(X)$ is a Fréchet space.*

PROOF. Whenever X is a k-space, $C(X)$ is complete; see, e.g., [9, p. 81]. Assume (1). Since $C(X)$ is separable, a theorem of Warner [13, p. 270] guarantees there is a separable metric topology T on X which is weaker than the given locally compact topology on X. Each compact set in the given topology of X is homeomorphic with itself in the T-topology and, consequently, is metrizable. Hence each compact subset is separable. Let (μ_n) be a total sequence in M (hypothesis (1)); if A_n is the compact support of μ_n, then $D = \bigcup_{n=1}^{\infty} A_n$ is dense in X. Since each A_n is separable, X is separable.

If p is any point in X and U is an open neighborhood of p with compact closure, then U is metrizable. Consequently, the topology on X has a countable base at p. It follows from separability that X has a countable basis for its locally compact topology and, therefore, is metrizable [7, p. 75]. Again by a theorem of Warner [13, p. 271] (it is also easy to see directly), $C(X)$ is metrizable.

If we assume X is a separable metric space, then $C(X)$ is a separable

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Fréchet space [13, p. 271]. This implies $\sigma(M, C)$ is separable [9, p. 165] and (1) is established.

A biorthogonal sequence $(x_n; f_n)$ in $E \times E'$ is a *Schauder basis* for E if, for each x in E, $\lim_{k\to\infty}\sum_{n=1}^{k} f_n(x)x_n = x$ for the given locally convex topology on E.

2. COROLLARY. *If $C(X)$ has a Schauder basis $(x_n; \mu_n)$, then (1) and (2) of the theorem above are satisfied.*

PROOF. The sequence $(\mu_n; x_n)$ is a $\sigma(M, C)$-Schauder basis for M.

The following theorem is established in [5].

3. THEOREM. *The following are equivalent*:

(1) *The strong topology $\beta(M, C)$ is separable.*

(2) *X is countable.*

We now strengthen the previous results by assuming the basis of $C(X)$ is unconditional. To do this we need the following definition and lemma. If S is the set of all finite subsets of the positive integers, then a series $\sum_k x_k$ is said to be *unconditionally convergent* when the net $\{\sum_\sigma x_k\colon k \in \sigma, \sigma \in S\}$ converges. A Schauder basis $(x_n; f_n)$ for E is called *unconditional* if for each x in X the series $\sum_n f_n(x)x_n$ converges unconditionally to x.

4. LEMMA. *Let E be a complete or barrelled space with a $\sigma(E', E'')$-sequentially complete dual E'. If $\sum_k f_k$ is a series in E', then this series is unconditionally convergent in $\beta(E', E)$ iff for each x in E, $\sum_{k=1}^{\infty}|f_k(x)| < \infty$.*

PROOF. Assume $\sum_{k=1}^{\infty}|f_k(x)| < \infty$ for each x in E. Then $F = \{\sum_\sigma f_k\colon \sigma \in S\}$ is $\sigma(E', E)$-bounded. It follows from [12, p. 72] or [9, p. 171] that F is $\sigma(E', E'')$-bounded.

Let $\sum_n f_{k_n}$ be a subseries of $\sum_k f_k$ and let $\phi \in E''$; then there is a constant $K(\phi) > 0$ such that $|\phi(\sum_\sigma f_k)| \leqslant K(\phi)$. Consequently, from a well-known fact about series of real numbers we have $\sum_{n=1}^{\infty}|\phi(f_{k_n})| \leqslant 2K(\phi) < \infty$. It follows that $\{\sum_n f_{k_n}\}$ is $\sigma(E', E'')$-Cauchy and thus convergent by hypothesis. By the Orlicz-Pettis Theorem [11, p. 297], $\sum_k f_k$ is $\beta(E', E)$-subseries convergent and consequently $\beta(E', E)$-unconditionally convergent.

The converse is clear.

Using a technique developed by Karlin [8, p. 983] for the Banach space $C[0, 1]$, we obtain the next theorem.

5. PROPOSITION. *Let $C(X)$ be barrelled, and let $(x_n; \mu_n)$ be an unconditional Schauder basis for $C(X)$. Then X is countable and $(\mu_n; x_n)$ is a $\beta(M, C)$-unconditional Schauder basis.*

PROOF. First, M is $\sigma(M, M')$-sequentially complete. To see this let (ν_n) be a $\sigma(M, M')$-Cauchy sequence. Since $C(X)$ is barrelled, there is a compact set $A \subset X$ such that the support of each $\nu_n \subset A$. Let M_A be the Banach space of all signed regular Borel measures on A. It is well known that this space is weakly sequentially complete [4, p. 108]. Next, $(\mu_n;\ x_n)$ is a $\sigma(M, C)$-unconditional basis and therefore, for each f in $C(X)$ and μ in M, $\Sigma_{n=1}^{\infty}|\mu(x_n)\mu_n(f)| < \infty$. By Lemma 4, $(\mu_n;\ x_n)$ is a $\beta(M, C)$-unconditional Schauder basis. It follows from Theorem 3 that X is countable.

A few examples are now in order. If $C(X)$ is an infinite dimensional Banach space with an unconditional basis, then X is homeomorphic to the interval space $[\alpha]$ for some ordinal $\alpha < \omega^{\omega}$ [10, p. 297]. If X is countable and discretely topologized, then $C(X)$ is isomorphic to the space s, the countable product of lines. This space is a nuclear space and the unit coordinate functions constitute an unconditional basis for this space. If X is countable, $C(X)$ need not have an unconditional basis. The space $C(\omega^{\omega})$ is such an example [1, p. 62] or [10, p. 297].

If a locally convex space E has a Schauder basis $(x_n;\ f_n)$ then this basis is called *absolute* if for each continuous seminorm ρ on E and each x in E, $\Sigma_{n=1}^{\infty}|f_n(x)|\rho(x_n) < \infty$. Certainly, an absolute basis is an unconditional basis and therefore, a reasonable question is: When does $C(X)$ have an absolute basis?

6. THEOREM. *Let $C(X)$ be complete and barrelled with an absolute basis $(x_n;\ \mu_n)$. Then $C(X)$ is a Fréchet space and isomorphic to a countable product of lines.*

PROOF. We have already observed that $(\mu_n;\ x_n)$ is a $\beta(M, C)$-Schauder basis. If we can show the basis is boundedly complete then $C(X)$ will be reflexive [3], and by the results of Warner [13, pp. 273–274] $C(X)$ will be a countable product of real lines. Thus, let (a_n) be a sequence of real numbers such that $B = \{\Sigma_{n=1}^{k} a_n x_n\colon\ k \in \omega\}$ is a bounded set in $C(X)$. If for each continuous seminorm ρ on $C(X)$ we can show $\Sigma_{n=1}^{\infty}|a_n|\rho(x_n) < \infty$, from the completeness of $C(X)$ we will have the existence of an x in $C(X)$ such that $\mu_n(x) = a_n$. The set $F = \{\Sigma_{\sigma}\rho(x_n)\mu_n\colon\ \sigma \in S\}$ is an equicontinuous set in E', since F is pointwise bounded. To see this let $x \in C(X)$, $|(\Sigma_{\sigma}\rho(x_n)\mu_n)(x)| \leqslant \Sigma_{n=1}^{\infty}|\mu_n(x)|\rho(x_n) < \infty$. From the equicontinuity of F and the boundedness of B, there exists $K > 0$ such that for each finite set σ and a sufficiently large index k,

$$\left|\sum_{\sigma} a_n\rho(x_n)\right| = \left|\left(\sum_{\sigma}\rho(x_n)\mu_n\right)\left(\sum_{n=1}^{k} a_n x_n\right)\right| \leqslant K.$$

As in the proof of Lemma 4, $|\Sigma_{\sigma} a_n\rho(x_n)| \leqslant K$ for all $\sigma \in S$ implies $\Sigma_{n=1}^{\infty}|a_n|\rho(x_n) \leqslant 2K < \infty$.

In the preceding proof the absolute convergence of the representation of each element of $C(X)$ was used only to show that the basis was boundedly complete. Therefore, we have the following

7. COROLLARY. *If $C(X)$ is complete and barrelled with an unconditional and boundedly complete basis then $C(X)$ is isomorphic to a countable product of real lines.*

We conclude with the following example. Let Q be the rational numbers in the unit interval with their ordinary topology. Then Q is metrizable but not locally compact, $C(Q)$ is a Fréchet space and not isomorphic to a countable product of lines. Does $C(Q)$ have an unconditional basis?

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