HOPF INVARIANTS
FOR REDUCED PRODUCTS OF SPHERES

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ABSTRACT. Let S_m^n be the mth reduced product complex of the even dimensional sphere S^n. Using 'cup'-products, James defined a Hopf invariant homomorphism

$$H_m^n: \pi_{mn-1}(S_{m-1}^n) \to \mathbf{Z}$$

such that H_2^n is the classical Hopf invariant. Extending the result of Adams on H_2^n we determine the image of H_m^n. Partial calculations were made by Hardie and Shar.

1. **Hopf invariants and higher order Whitehead products.** In [9] James proved, that the reduced product complex S_∞^n of the sphere S^n is homotopy equivalent to the loop space ΩS^{n+1}. The complex S_∞^n has a natural CW decomposition

$$S_\infty^n = S^n \cup e^{2n} \cup \ldots \cup e^{mn} \cup \ldots$$

the mn skeleton of which is denoted by S_m^n, $S_1^n = S^n$. In this paper we will be concerned with the homomorphism

$$H_m^n: \pi_{mn-1}(S_{m-1}^n) \to \mathbf{Z}$$

where n is even and $m \geqslant 2$. This homomorphism is a generalization due to James [10] of Steenrod's definition of the Hopf invariant [21]: Let $\alpha \in \pi_{mn-1}(S_{m-1}^n)$; one can choose generators a_1, a_{m-1} and x of dimension n, $(m-1)n$ and mn respectively in the integral cohomology of the complex $S_{m-1}^n \cup_\alpha E^{mn}$. Then the Hopf invariant $H_m^n(\alpha)$ is defined to be the integer for which $a_1 \cup a_{m-1} = H_m^n(\alpha)x$. H_2^n is the classical Hopf invariant [8].

Let the element $[i_n]^m \in \pi_{mn-1}(S_{m-1}^n)$ be given by an attaching map of the cell e^{mn} in S_m^n. This element is an mth order Whitehead product [16], [6]. For example $[i_n]^2$ is the Whitehead product $[i_n, i_n]$ of a generator $i_n \in \pi_n(S^n)$. It is well known that $H_2^n([i_n, i_n]) = 2$ and more generally $H_m^n([i_n]^m) = m$; see §3.

On the other hand we have $H_2^n(\sigma_n) = 1$ for the Hopf elements $\sigma_n \in \pi_{2n-1}(S^n)$, $n = 2, 4, 8$ [8] and im $H_2^n = 2\mathbf{Z}$ if $n \neq 2, 4, 8$ by the celebrated theorem of Adams on Hopf invariants [2]. Moreover Toda showed in [23, p. 175] that for a prime number p, there exist $\alpha_p \in \pi_{2p-1}(S_{p-1}^2)$ (and $\alpha_2 = \sigma_2$ if $p = 2$) such that $H_p^2(\alpha_p) = 1$. We will prove that the elements α_p and the Hopf elements σ_2, σ_4, σ_8 are the only elements of Hopf invariant one.

THEOREM A.

$$\operatorname{im} H_m^n = \begin{cases} \mathbf{Z}, & \textit{if } m = 2 \textit{ and } n = 2, 4, 8, \\ \mathbf{Z}, & \textit{if } n = 2 \textit{ and } m \textit{ a prime number,} \\ m\mathbf{Z}, & \textit{otherwise.} \end{cases}$$

ACKNOWLEDGEMENT. It follows easily from (1.3), (c), (ii) of [6] that $\operatorname{im} H_m^n = m\mathbf{Z}$ if $n \geqslant 4$ and m is an odd prime. Using this result Shar showed that $\operatorname{im} H_m^n = m\mathbf{Z}$ for $n \geqslant 4$ and $m \geqslant 3$ [19], [20]. Thus we only have to prove $\operatorname{im} H_m^2 = m\mathbf{Z}$ in case m is not a prime. Using our method of proof one can also deduce Shar's result. Our proof is nevertheless different from Shar's proof, since in the case of the reduced product of a 2-sphere we cannot use Lemma 2.1 in [19].

From the exact sequence of the pair (S_∞^n, S_{m-1}^n) we have a short exact sequence:

$$(1.1) \qquad 0 \to \pi_{mn}(S_\infty^n, S_{m-1}^n) \xrightarrow{\partial} \pi_{mn-1}(S_{m-1}^n) \xrightarrow{\iota} \pi_{mn}(S^{n+1}) \to 0.$$

There is a generator ω_m^n of $\pi_{mn}(S_\infty^n, S_{m-1}^n) \cong \mathbf{Z}$ with $\partial\omega_m^n = [i_n]^m$. Therefore $\iota(\alpha)$ is a nontrivial element if $0 < H_m^n(\alpha) < m$. For example $\iota(\alpha_p)$ is the well-known element of order p in $\pi_{2p}(S^3)$ and $\iota(\sigma_n)$ is the nontrivial suspension of the Hopf element. By Theorem A, this method will not yield further nontrivial elements in the homotopy groups of spheres. On the other hand the short exact sequence above immediately implies the following corollary of Theorem A:

(1.2) COROLLARY. *In case* $\operatorname{im} H_m^n = m\mathbf{Z}$, *we have* $\pi_{mn-1}(S_{m-1}^n) = \mathbf{Z} \oplus \pi_{mn}(S^{n+1})$ *and* $[i_n]^m$ *generates the infinite cyclic summand.*

Using this corollary we will determine the order of the Whitehead products $[i_n, [i_n]^{m-1}] \in \pi_{mn-2}(S_{m-2}^n)$, where $i_n \in \pi_n(S_{m-2}^n)$ is a generator and $m > 2$. That $m[i_n, [i_n]^{m-1}] = 0$, is a special case of the Jacobi identity for the higher order Whitehead products (Hardie [5], [6]). Moreover Hardie has proven in [6] that $[i_2, [i_2]^{m-1}] = 0$ if m is a prime. In fact:

(1.3) COROLLARY. $[i_n, [i_n]^{m-1}] = 0$ *if and only if* $n = 2$ *and* m *is a prime; and* $[i_n, [i_n]^{m-1}]$ *is an element of order* m *otherwise.*

If $m = 3$ this is a well-known result on the iterated Whitehead product $[i_n, [i_n, i_n]] \in \pi_{3n-2}(S^n)$; cf. [15], [13]. Shar has proven the result of Corollaries (1.2) and (1.3) in case $n \geqslant 4$ [19]. Compare also Hardie's result (0.2)(b) in [7].

PROOF OF (1.3). We consider the exact sequence

$$\pi_{mn-1}(S_{m-1}^n) \xrightarrow{j} \pi_{mn-1}(S_{m-1}^n, S_{m-2}^n) \xrightarrow{\partial} \pi_{mn-2}(S_{m-2}^n).$$

Using the Nakaoka Toda formula [5], [6] we have $j([i_n]^m) = m[i_n, \omega_{m-1}^n]$. The relative Whitehead product $[i_n, \omega_{m-1}^n]$ generates the infinite cyclic part of $\pi_{mn-1}(S_{m-1}^n, S_{m-2}^n)$, see 1.4, p. 262 in [11]. Since $\partial[i_n, \omega_{m-1}^n] = \pm[i_n, [i_n]^{m-1}]$, the result follows using (1.2).

2. **The chain algebra of** ΩS_∞^n. We obtain our result on the Hopf invariant H_m^n by examining $H_{mn-2}(\Omega S_{m-1}^n)$, with integral coefficients. Let the homomorphism

$$\tau \colon \pi_{mn-1}(S_{m-1}^n) \cong \pi_{mn-2}(\Omega S_{m-1}^n) \to H_{mn-2}(\Omega S_{m-1}^n)$$

be given by composition with the Hurewicz homomorphism as in [17]. We
will prove:

THEOREM B. *There is a homomorphism* $T\colon H_{mn-2}(\Omega S_{m-1}^n) \to \mathbf{Z}$ *such that*

$$T\tau([i_n]^m) = \begin{cases} p, & \text{if } m = p^v \text{ is a power of a prime } p, \\ 1, & \text{otherwise.} \end{cases}$$

(2.1) COROLLARY. *If m is not a power of a prime number, then* im $H_m^n = m\mathbf{Z}$.
If $m = p^v$ is a power of the prime p, then im $H_m^n = m\mathbf{Z}$ *or* im $H_m^n = p^{v-1}\mathbf{Z}$.

PROOF OF (2.1). Let $\alpha \in \pi_{mn-1}(S_{m-1}^n)$ be given such that im $H_m^n = H_m^n(\alpha)\mathbf{Z}$.
By use of the exact sequence (1.1) we obtain $\pi_{mn-1}(S_{m-1}^n) \cong \mathbf{Z} \oplus T$ (T a
torsion group) and α generates the infinite cyclic summand. Since $[i_n]^m$
$= k\alpha + t$ ($t \in T$) we have $m = kH_m^n(\alpha)$. On the other hand k is a divisor of
$T\tau([i_n]^m)$, and so the corollary follows from Theorem B.

For the proof of Theorem B we need the chain algebra of ΩS_∞^n in the sense
of [1]. Let $A = A(S_\infty^n) = A[x_1, x_2, \dots]$ be the unitary ring, which is freely
generated by the elements $x_1, x_2, \dots$ and graded by $\deg(x_i) = ni - 1$. We
define a differential $d\colon A \to A$ on generators by

$$(2.2) \qquad\qquad dx_i = \sum_{j=1}^{i-1} \binom{i}{j} x_j x_{i-j}.$$

For products ab in A we define $d(ab) = (da)b + (-1)^k a(db)$, where $k = \deg(a)$. Let $A^m = A[x_1, \dots, x_m]$ be the chain subalgebra of A generated by
$x_1, \dots, x_m$. Then we obtain from [3] and [14]:

(2.3) LEMMA. *There exists an isomorphism* $\Theta\colon H_*(A^{m-1}, d) \to H_*(\Omega S_{m-1}^n)$,
such that $\Theta\{dx_m\} = \tau([i_n]^m)$. *Here dx_m is a cycle in A^{m-1} representing the
nonzero homology class* $\{dx_m\}$.

(2.4) PROOF OF THEOREM B. Let C_k be the kth chain group of the chain
complex A^{m-1}, $k \geq 0$. Let $X_i \subset C_{nm-i}$ be generated by monomials in A^{m-1}
with at most i factors, $i = 1, 2, 3$. Let $Y_i \subset C_{nm-i}$ be generated by monomials in
A^{m-1} with more than i factors. By (2.2) we obtain restrictions d_j' and d_j''
($j = 1, 2$) of d such that the following diagram commutes:

$$
\begin{array}{ccccc}
C_{nm-1} & = & X_1 & \oplus & Y_1 \\
d\downarrow & & & \downarrow d_1' \oplus d_2' & \\
C_{nm-2} & = & X_2 & \oplus & Y_2 \\
d\downarrow & & & \downarrow d_1'' \oplus d_2'' & \\
C_{nm-3} & = & X_3 & \oplus & Y_3
\end{array}
$$

Since $X_1 = 0$ we obtain by (2.3) the isomorphism

$$\Theta^{-1}\colon H_{nm-2}(\Omega S_{m-1}^n) \to \ker d_1'' \oplus \ker d_2'' / \mathrm{im}\, d_2'.$$

A simple computation using (2.2) shows that ker $d_1'' \cong \mathbf{Z}$ and is generated by
dx_m/g_m, where $g_m = \gcd\{\binom{m}{i}|0 < i < m\}$. Let T be given by composition of
Θ^{-1} with projection on ker d_1''. Then we have $T\tau([i_n]^m) = g_m$ by (2.3). One
can check that $g_m = p$ if $m = p^v$ is a power of the prime p and that $g_m = 1$
otherwise. This completes the proof of Theorem B.

3. **Proof of Theorem A.** Let $\alpha \in \pi_{mn-1}(S^n_{m-1})$ and let the complex $K = S^n_{m-1} \cup_\alpha E^{mn}$ be obtained by attaching an oriented mn cell E^{mn} to S^n_{m-1} by a map f of homotopy class α. Using the result of Serre on the cohomology of S^n_∞, we can choose generators $a_i \in H^{ni}(K)$, where $1 \leqslant i < m$, such that

$$(1) \qquad\qquad\qquad a_1^i = i! a_i,$$

cf. Theorem 18 on p. 488 in [18]. Let the Hopf invariant $H^n_m(\alpha)$ be defined by use of these generators. In case $\alpha = [i_n]^m$ and $K = S^n_m$ this implies $H^n_m([i_n]^m) = m$.

For the proof of Theorem A we need only show that the assumption $\operatorname{im} H^2_m = p^{\iota-1}\mathbf{Z}$ leads to a contradiction if $m = p^\iota$ is a prime power with $\iota > 1$; see (2.1). Assuming this, it follows that there exists an element α such that $H^2_m(\alpha) = p^{\iota-1}$. Thus in K we have

$$(2) \qquad\qquad\qquad a_1 \cup a_{m-1} = p^{\iota-1}x.$$

Multiplying by $(m-1)!$, we obtain from (1):

$$(3) \qquad a_1^m = a_1 \cup (m-1)! a_{m-1} = (p^\iota - 1)! p^{\iota-1}x = p^\iota! p^{-1}x.$$

Thus if $q = p^{\iota-1}$, we have $qp = m$, and

$$(4) \qquad\qquad\qquad a_1^m = (a_1^q)^p = (q! a_q)^p = (q!)^p a_q^p.$$

Therefore we obtain $a_q^p = r_p x$, where

$$(5) \qquad\qquad\qquad r_p = (p^{\iota-1}!)^{-p}(p^\iota!)p^{-1}.$$

Let $a_{q,p} \in H^{2q}(K, \mathbf{Z}_p)$ and $x_p \in H^{2m}(K, \mathbf{Z}_p)$ be generators, which correspond to a_q and x in case of integral coefficients. Since $r_p \not\equiv 0 \bmod p$, we have:

$$(6) \qquad\qquad\qquad (a_{q,p})^p = r_p x_p \neq 0.$$

The following arguments generalize the method of Nakaoka and Toda on p. 12 in [15] and the method of Hardie on p. 247 in [6]: Since the suspension Σ of a higher order Whitehead product is trivial, there exists a mapping ρ: $\Sigma S^2_{m-1} \to S^{2q+1}$ which induces an isomorphism of cohomology in dimension $2q+1$. Let $g = \rho \circ (\Sigma f)$, where $f \in \alpha$. Then we have an extension $\bar\rho$: $\Sigma K \to S^{2q+1} \cup_g E^{2m+1} = K_g$ such that $\bar\rho$ induces isomorphism of cohomology in dimension $2q+1$ and $2m+1$.

We examine the cases p odd and $p = 2$ separately. If p is odd, it follows from (6) that the Steenrod pth power

$$(7) \qquad\qquad\qquad P^q(a_{q,p}) = (a_{q,p})^p \neq 0.$$

By the naturality of the reduced power operation (using $\bar\rho$) and the fact that they commute with suspension, we obtain from (7) an isomorphism:

$$(8) \qquad\qquad\qquad P^q: H^{2q+1}(K_g, \mathbf{Z}_p) \to H^{2m+1}(K_g, \mathbf{Z}_p).$$

Hence the mod p Hopf invariant of g is nontrivial. Now by Theorem 5 of [13] this is only the case if $q = 2$ and so we have the required contradiction.

On the other hand if $p = 2$, it follows from (6) that the Steenrod square

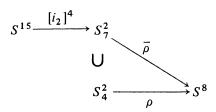$$(9) \qquad\qquad\qquad Sq^{2q}(a_{q,2}) = (a_{q,2})^2 = r_2 x_2 \neq 0.$$

Since the suspension $\Sigma_*\colon \pi_{2m-1}(S^{2q}) \to \pi_{2m}(S^{2q+1})$ is an epimorphism by the Freudenthal theorem, we have a mapping $\bar{g}\colon S^{2m-1} \to S^{2q}$ such that $\Sigma \bar{g}$ and g are homotopic. We also obtain a homotopy equivalence $t\colon \Sigma K_{\bar{g}} \to K_g$. By the naturality of the Steenrod squares (using $\bar{p}$ and t) and the fact that they commute with suspension we obtain from (9) an isomorphism:

(10) $$\mathrm{Sq}^{2q}\colon H^{2q}(K_{\bar{g}}, \mathbf{Z}_2) \to H^{2m}(K_{\bar{g}}, \mathbf{Z}_2).$$

Hence the Hopf invariant $H_2^{2q}(\bar{g})$ is odd. Since this is only the case if $m = 2$, 4, 8 we have the required contradiction for $m = 2^v$ and $v > 3$; cf. 4.5 in [22] and [2]. Thus we still have to find a contradiction in case $m = 4$ and $m = 8$.

In case $m = 4$ we assume that there exists an α such that $H_4^2(\alpha) = 2$. By use of the exact sequence (1.1), $\iota(\alpha) \in \pi_8(S^3)$ is a nontrivial element. Using the exact sequence in the proof of (1.3), we know that $j(\alpha)$ is nontrivial and thus $\iota(\alpha)$ is an element of filtration 3. Since James has proven that all elements of $\pi_8(S^3)$ have filtration less than 3, we have a contradiction; cf. row 3, p. 309 in [12].

In case $m = 8$ we consider the following diagram:

where ρ is a mapping which induces an isomorphism of homology in dimension 8.

In [4] we prove that there exists an extension $\bar{\rho}$ of ρ such that

(11) $$\bar{\rho}_*([i_2]^4) = 35[i_8, i_8] \in \pi_{15}(S^8).$$

If we assume that im $H_8^2 = 4\mathbf{Z}$, then there exists an $\alpha \in \pi_{15}(S_7^2)$ such that $H_8^2(\alpha) = 4$ and such that $[i_2]^4 = 2\alpha + \iota$; compare the proof of (2.1). Since $\pi_{16}(S^3) = \mathbf{Z}_6$, the element ι has order at most 3, using the exact sequence (1.1). From Toda's book [23] we know that $\pi_{15}(S^8) = \mathbf{Z} + \mathbf{Z}_{120}$, and the Hopf element σ_8 generates the infinite cyclic summand. Let σ be the generator of $\mathbf{Z}_{120}$. Then we have $[i_8, i_8] = \pm(2\sigma_8 - 15\sigma)$; see 5.16 on p. 50 in [23]. Now there exist $w, a, b \in \mathbf{Z}$ such that $\bar{\rho}_*(\iota) = 40 \cdot w\sigma$ and $\bar{\rho}_*(\alpha) = a\sigma_8 + b\sigma$. Thus we obtain from (11):

(12) $$\pm 35(2\sigma_8 - 15\sigma) = 2(a\sigma_8 + b\sigma) + 40 \cdot w\sigma.$$

Since $35 \cdot 15 \not\equiv 2b + 40w \bmod 120$ for $b, w \in \mathbf{Z}$, we have the required contradiction and the proof of Theorem A is complete.

As pointed out by the referee in case $m = 8$ a filtration argument is as well available, since the reduced product filtration of the generator of $\pi_{16}(S^3, 2)$ is 2.

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