REMARKS ON THE CARDINALITY OF COMPACT SPACES
AND THEIR LINDELÖF SUBSPACES

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Abstract. Several applications of the Čech-Pospíšil theorem are given; one of them states (under CH) that every uncountable compact space has a Lindelöf subspace of cardinality $\omega_1$.

Notation and definitions are the same as in [1], except that a $G_\kappa$ set means a set which is the intersection of $\leq \kappa$ open sets. Thus if $\kappa = \omega_2$, then $G_\kappa$ is the same as $G_{\omega_2}$ in [1].

Let us recall at the outset the old theorem of Čech and Pospíšil which states that if $X$ is compact $T_2$ and $\chi(p, X) > \kappa > \omega$ hold for each $p \in X$, then $|X| > 2^\kappa$. In [1] and [2] it is (incorrectly) stated that this theorem remains valid if we only require $X$ to be a $G_\kappa$ subset of some compact $T_2$ space. This is not true; however, it turns out that if we replace $G_\kappa$ with $G_\lambda$ for some $\lambda < \kappa$, then a simple argument can reduce it to the Čech-Pospíšil theorem.

**Proposition 1.** If $X \subseteq Z$ is a (nonempty) $G_\lambda$-subset with $\lambda < \kappa$, where $Z$ is compact Hausdorff, and $\chi(p, X) > \kappa$ for each $p \in X$, then $|X| > 2^\kappa$.

**Proof.** It is known (cf. [1, 2.11]) that $X$ contains a nonempty subset $Y$ which is a closed $G_\kappa$-subset of $Z$. Thus $Y$ is compact and we claim that $\chi(p, Y) > \kappa$ for each $p \in Y$. Indeed, since $\psi(p, Y) = \chi(p, Y)$, otherwise we would have $\psi(p, Z) = \chi(p, Z) < \kappa$, considering that $Y$ is a $G_\kappa$-subset of $Z$, contradicting that $\kappa < \chi(p, X) < \chi(p, Z)$. Hence we have $2^\kappa < |Y| < |X|$.

The following result is not proved directly from the Čech-Pospíšil theorem but from its proof. It seems to be new and will be crucial in the proof of Theorem 1.

**Proposition 2.** If $X$ is a compact Hausdorff space with $\chi(p, X) > \kappa$ for all $p \in X$, then there exists a subset $C \subseteq X$ with $|C| < 2^\kappa = \Sigma(2^\lambda: \lambda < \kappa)$ such that $|C| > 2^\kappa$.

**Proof.** The basic idea in all known proofs of the Čech-Pospíšil theorem is the construction of a certain kind of ramification system of closed sets. To be more precise, for every $0$-$1$ sequence $\varepsilon$ of length $\nu < \kappa$ one defines a nonempty closed set $X_\varepsilon \subseteq X$ such that

(i) if $\varepsilon$ is a limit ordinal, then $X_\varepsilon = \bigcap\{X_{\varepsilon[\mu]}: \mu < \nu\}$;
(ii) $X_0, X_1 \subseteq X_\varepsilon$ and $X_0 \cap X_1 = \emptyset$.

Let $S_\nu$ denote the set of all $0$-$1$ sequences of length $\nu$. For every $\nu < \kappa$, $\varepsilon \in S_\nu$ and $i < 2$, the set $X_\varepsilon \setminus X_{\varepsilon[i]}$ is nonempty; hence we can choose for each
such $\varepsilon$ and $i$ a point $p_{\varepsilon,i} \in X_\varepsilon \setminus X_{\eta_i}$. Now put

$$C = \{ p_{\varepsilon,i} : \varepsilon \in \cup \{ S_\nu : \nu < \kappa \} & i \in 2 \}. $$

It is obvious that $|C| \leq 2^\kappa$. Next we show that for each $\eta \in S_\kappa$ we have $X_\eta \cap C \neq \emptyset$; this will obviously imply $|C| \geq 2^\kappa$. To see this, let

$$C_\eta = \{ p_{\eta,\nu} : \nu < \kappa \} \subset C. $$

(For simplicity we write $p_{\eta,\nu} = p^{(\nu)}$. Then $\mu < \nu < \kappa$ implies $p^{(\mu)} \neq p^{(\nu)}$, hence $|C_\eta| = \kappa$. Since $X$ is compact, $C_\eta$ must have a complete accumulation point in $X$, and we claim that any such point must belong to $X_\eta$. This will imply $C \cap X_\eta \supseteq C_\eta \cap X_\eta \neq \emptyset$. Indeed, let $q \in X \setminus X_\eta$ arbitrary. Then there is a $\nu < \kappa$ such that $q \in X \setminus X_{\eta,\nu}$, but obviously, this latter set is an open neighbourhood of $q$, whose intersection with $C_{\eta}$ is $\{ p^{(\mu)} : \mu < \nu \}$, hence of cardinality less than $\kappa$. This completes the proof.

The next result was obtained while trying to solve the following problem: Is it true that a Lindelöf space of cardinality $\omega_2$ must contain a Lindelöf subspace of cardinality $\omega_1$? (GCH is assumed.) A natural thing was to try it for compact spaces first.

**Theorem 1.** Assume the CH (i.e. $2^{\omega} = \omega_1$). Then every uncountable compact space has a Lindelöf subspace of cardinality $\omega_1$.

Before we can start the actual proof, we state a lemma, which is interesting in itself.

**Lemma.** If a space $X$ has a point $p \in X$ with $\chi(p, X) = \omega_1$, then $X$ contains a Lindelöf subspace of cardinality $\omega_1$.

**Proof.** Let $\{ U_\mu : \mu < \omega_1 \}$ be a basis of neighbourhoods for $p$ in $X$. We define points $p_\mu$ for $\mu < \omega_1$ by transfinite induction, as follows. Suppose that $\mu < \omega_1$ and for each $\nu < \mu$ the point $p_\nu$ has already been defined.

Then we choose $p_\mu$ in $\cap \{ U_\nu : \mu < \nu \} \setminus (\{ p_\nu : \mu < \nu \} \cup \{ p \})$, which is possible by $\chi(p, X) = \omega_1$.

Now let us put $S = \{ p_\mu : \mu < \omega_1 \} \cup \{ p \}$ and let $G$ be any neighbourhood of $p$ in $S$. Then there is a $\mu < \omega_1$ such that $G \supset U_\mu \cap S$. But obviously $U_\mu \cap S = \{ p_\nu : \nu < \mu < \omega_1 \}$, hence $|S \setminus G| < |S \setminus U_\mu| < \omega$, i.e. any open cover of $S$ contains a member, whose complement is countable; consequently $S$ must be Lindelöf.

**Proof of Theorem 1.** Now let $X$ be an uncountable compact space, and let $A \subset X$, $|A| = \omega_1$. If $|\bar{A}| = \omega_1$, too, we are done. Hence we assume $|\bar{A}| > \omega_1$.

If $p \in \bar{A}$ and $\chi(p, \bar{A}) = \omega$, then we can choose a sequence from $A$ converging to $p$, hence

$$\left| \{ p \in \bar{A} : \chi(p, \bar{A}) < \omega \} \right| \leq \omega^\omega \leq \omega_1 \quad \text{(by CH).}$$

Applying the lemma, we see that if $\chi(p, \bar{A}) = \omega_1$, for some $p \in \bar{A}$, then we are again done, hence we might assume $\chi(p, A) \neq \omega_1$ for all $p \in A$. Consequently if $p \in \bar{A}$ is such that it has a neighbourhood $U$ (in $\bar{A}$) with $|U| < \omega_1$, then we have $\psi(p, A) = \chi(p, A) < \omega_1$, hence $\chi(p, A) \leq \omega$. Now let $G$ be the union of all open subsets of $\bar{A}$ of cardinality $\leq \omega_1$. Then from what we proved above, for every point $p$ of $G$ we have $\chi(p, A) \leq \omega$, hence $|G| \leq \omega_1$. By the definition of $G$, $\bar{A} \setminus G$ is compact and has no isolated
points (i.e. $\chi(p, A \setminus G) \geq \omega$ for all $p \in A \setminus G$). By Proposition 2, there is a set $C \subset A \setminus G$ with $|C| = \omega$ and $|\bar{C}| > \omega_1$. Again, we might assume that $|\bar{C}| > \omega_1$.

Since $\bar{C}$ has a countable dense subset, we have $w(\bar{C}) < 2^\omega = \omega_1$; consequently $\chi(p, \bar{C}) < \omega_1$, for all $p \in \bar{C}$.

Moreover in the same way as above it follows that

$$|\{p \in \bar{C}: \chi(p, \bar{C}) < \omega\}| < \omega_1.$$

Hence there is a $p \in \bar{C}$ with $\chi(p, \bar{C}) = \omega_1$, and thus, by the lemma, there is a Lindelöf subspace of cardinality $\omega_1$.

**Remark.** A similar argument can be used to show that, assuming GCH, for any $\kappa$ and any compact space of cardinality $> \kappa$ there is a $\kappa$-Lindelöf subspace of cardinality $\kappa^+$. It should be interesting to decide whether the first $\kappa$ in the conclusion could be lowered (perhaps even to $\omega$).

It is natural to raise the question why we have not asked about the existence of compact (i.e. closed) subspaces of cardinality $\omega_1$ of uncountable compact spaces. The space $\beta N$, however, shows that the answer to this question is negative as it is well known that any infinite closed subspace of $\beta N$ has the cardinality $2^\omega$.

Our next result shows that compact spaces without small infinite closed subsets must be large; under CH they have to have the same cardinality as $\beta N$.

**Theorem 2.** Suppose $X$ is compact and every infinite closed subset of $X$ is uncountable. Then $|X| \geq 2^\omega$.

**Proof.** Let $C$ be the set of all nonisolated points of $X$. First we claim that no point of $C$ is isolated in $C$, either. Indeed, let $p \in C$ and assume, on the contrary, that $U$ is a closed neighbourhood of $p$ in $X$ such that $U \cap C = \{p\}$. However then $U$ is infinite, because $p$ is not isolated in $X$, and for any countably infinite $A \subset U$ the set $A \cup \{p\}$ is closed because its complement in $U$ consists of isolated points only. Thus $C$ must be dense in itself.

However, for no point $p \in C$ can we have $\chi(p, C) = \omega$, because in this case we could select from $C \setminus \{p\}$ an ordinary sequence $\langle p_n: n \in \omega \rangle$ of different points converging to $p$, and then $\{p_n: n \in \omega\} \cup \{p\}$ would be a countably infinite closed subset of $C$, and thus of $X$. But then for each $p \in C$ we have $\chi(p, C) > \omega_1$, consequently, by the Čech-Pospíšil theorem, $|X| > |C| > 2^\omega$.

**References**


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