ON MAXIMALITY OF GORENSTEIN SEQUENCES

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Abstract. It is well known that if $A$ is a Gorenstein ring, then every ideal generated by a regular sequence $x \subseteq A$ has irreducible (minimal) primary components. This feature led us to define a Gorenstein sequence of a ring $A$ to be any ordered regular sequence $x = \{x_1, \ldots, x_r\} \subseteq A$ such that for every $i \in \{1, \ldots, r\}$ the ideal $(x_1, \ldots, x_i)$ has irreducible minimal primary components. We showed for Gorenstein sequences (G-sequences for short) some parallels of well-known properties of regular sequences and moreover by means of G-sequences we gave the following natural characterization of local Gorenstein rings: "A local ring $(A, m)$ is Gorenstein iff $m$ contains a G-sequence of length $= K - \dim A$ ."

In this note we are going to give some information about "maximality" of G-sequences in a local ring $A$, producing sufficient conditions on $A$ in order that the maximal G-sequences of $A$ all have the same length, i.e. in order to give a "good" definition of G-depth $A$. Furthermore, we will state some results about the G-depth behavior with respect to local flat ring homomorphisms.

0. Notations and preliminary results. Throughout this paper $A$ will denote a commutative noetherian ring with unity. In addition, we will say that the Gorenstein locus of $A$ is open if there exists an ideal $\mathfrak{J} \subseteq A$ such that for every $p \in \text{Spec } A$, $A_p$ is a local Gorenstein ring iff $p \supseteq \mathfrak{J}$. In [H. K., Vortrag 6] it has been shown that the Gorenstein locus of a local Cohen-Macaulay (C. M. for short) ring $A$ is open if there exists the canonical module $K_A$ (for the definition and properties of $K_A$ see [H. K., Vortrag 5] where it has also been given a sufficient criterion for the existence of $K_A$). If $K_A$ exists then the ideal $\mathfrak{J} \subseteq A$ which defines the non-Gorenstein locus of $A$ admits an explicit description (see [H. K., Bemerkung and Lemma 6.19]) as the ideal generated by the elements $\varphi(x)$ with $x \in K_A$, $\varphi \in \text{Hom}_A(K_A, A)$, i.e. $\mathfrak{J}$ is the trace ideal of $K_A$ (cf. [De M. I., Chapter 1, §1B]).

In this connection we have the following handy characterization:

0.1. If $A$ is a ring with open Gorenstein locus and $\mathfrak{J} \subseteq A$ is the ideal defining the non-Gorenstein locus of $A$, then an ordered regular sequence $\{x_1, \ldots, x_r\} \subseteq A$ is a G-sequence iff for every $i \in \{1, \ldots, r\}$ and $p \in \text{Ass}(A/(x_1, \ldots, x_i))$ we have $p \supseteq \mathfrak{J}$.

Now we recall explicitly two concepts which will be frequently used in the following.

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0.2. The $G$-depth of a local ring $(A, m)$ is the largest number of elements in $m$ making up a $G$-sequence of $A$. If $m$ does not contain any $G$-sequence of positive length, we define the $G$-depth $A$ to be 0 if the zero ideal of $A$ has irreducible minimal primary components (i.e. the empty set $\emptyset \subseteq A$ makes up a $G$-sequence of length 0), otherwise $G$-depth $A = -\infty$ (cf. [M$_1$, Definition 4.1]).

0.3. A ring $A$ is said to be $G_n$ if $G$-depth $A_p \geq \min(n, \text{ht} p)$ for every $p \in \text{Spec } A$ (cf. [M$_1$, Definition 4.5]).

In [M$_2$, Theorem 2.1] we gave several equivalent characterizations for the $G_n$ condition, showing in particular that a ring $A$ is $G_n$ iff it is $S_n$ (i.e. depth $A_p \geq \min(n, \text{ht} p)$ for every $p \in \text{Spec } A$) and $A_p$ is a local Gorenstein ring for every $p \in \text{Spec } A$ such that $\text{ht} p < n$. Here we can add the following:

0.4. If $A$ is an $S_n$ ring with open Gorenstein locus and $\mathfrak{S} \subseteq A$ is the ideal defining the non-Gorenstein locus of $A$, then $A$ is $G_n$ iff $\text{ht} \mathfrak{S} > n$.

1. Principal results and examples.

**Proposition 1.1.** Let $A$ be any local C. M. ring with open Gorenstein locus and let $\mathfrak{S} \subseteq A$ be the ideal which defines the non-Gorenstein locus of $A$; then:

(a) if $\text{ht} \mathfrak{S} = 0$, $A$ does not contain any $G$-sequence;

(b) if $\text{ht} \mathfrak{S} > 0$, all maximal $G$-sequences of $A$ have the same length (exactly equal to $K - \text{dim } A$ or $K - \text{dim } A - 1$, according as $A$ is Gorenstein or not).

**Proof.** (a) $A$ is not a $G_0$ ring (cf. 0.4) so it is clear that $A$ does not contain any $G$-sequence of length $= 0$ (cf. [M$_2$, Osservazione (iii)]). On the other hand one can easily see that $A$ does not even contain $G$-sequences of positive length since if there existed $x \subseteq A$ a $G$-sequence of length $= r > 0$, then both $A/(x)$ and $A$ would be $G_0$ rings (respectively by [M$_2$, Osservazione (iv)] and [M$_3$, Lemma 2.2]) and this would clearly give a contradiction.

(b) First of all we observe that if $A$ is a Gorenstein ring, then manifestly all maximal $G$-sequences of $A$ must have the same length exactly equal to $K - \text{dim } A$ since it follows directly from the definitions that all regular sequences of any (not necessarily local) Gorenstein ring are also $G$-sequences (cf. [B, Fundamental Theorem], and [M$_1$, Definition 2.1]). Therefore to complete our proof we have only to examine the case $\mathfrak{S} \not\subseteq A$. Here $\text{ht} \mathfrak{S} > 0$ implies that for every $p \in \text{Spec } A$ such that $\text{ht} p = 0$, $A_p$ is a local Gorenstein ring (cf. 0.4): hence all (minimal) primary components of the zero ideal in $A$ are irreducible, i.e. the empty set $\emptyset \subseteq A$ is a $G$-sequence of length $= 0$ (cf. [M$_2$, Osservazione (vi)]). This just proves our theorem in case $K - \text{dim } A = 1$ where $\emptyset$ is actually the only (maximal) $G$-sequence of $A$ (we are assuming $A$ is not Gorenstein), so from now on we can suppose $K - \text{dim } A > 1$. Since $A$ is a C. M. ring $\text{ht} \mathfrak{S} > 0$ implies also that there exists some element $i \in \mathfrak{S}$ which is regular in $A$ and so can be completed to a maximal regular sequence of $A$. say $x = \{i, x_2, \ldots, x_m\}$ with $m = K - \text{dim } A$. We want to show that $x' = \{x_2, \ldots, x_m\}$ is a $G$-sequence of $A$; in this connection to say that $x \subseteq A$ is a regular sequence means in particular that $i \not\in \mathfrak{P}'$ for any $\mathfrak{P}' \in \text{Ass}(A/(x'))$, i.e. $A_{\mathfrak{P}'}$ is a local Gorenstein ring for every $\mathfrak{P}' \in \text{Ass}(A/(x'))$ and this means that $x'$ is a $G$-sequence, so $\emptyset$ is not a maximal $G$-sequence in $A$. Then let $y \subseteq A$ be any maximal $G$-sequence of positive length $= s < m$. 

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(recall that we are assuming \( A \) is not Gorenstein): for every \( \mathfrak{Q} \in \text{Ass} (A/(y)) \) clearly \( \mathfrak{Q} \supseteq \mathfrak{Q} \) so there exists some element \( j \in \mathfrak{Q} \) which is a non-zero-divisor modulo \( (y) \) such that \( \{y,j\} \) is a regular sequence of length \( = s + 1 \). If \( s < m - 1, \{y,j\} \) can be completed to a maximal regular sequence of \( A \), say \( \{y,j, y_{s+2}, \ldots, y_m\} \); here applying the same argument as above we can see that \( \{y, y_{s+2}, \ldots, y_m\} \) is a G-sequence of \( A \) containing \( y \), contradicting the maximality of \( y \), so all maximal G-sequences of \( A \) actually have length \( = K - \dim A - 1 \).

**Corollary 1.2.** If \( A \) is a local C. M. ring with open Gorenstein locus, then \( \text{G-depth} A \) is the length of any maximal G-sequence in \( A \) (as usual \( \text{G-depth} A = -\infty \) if \( A \) does not contain G-sequences of any length).

**Corollary 1.3.** Let \( A \) be any \( S_n \) local ring with \( K - \dim A > n > 1 \). If the Gorenstein locus of \( A \) is open, then either \( A \) does not contain any G-sequence or all maximal G-sequences of \( A \) have length \( \geq n - 1 \).

**Proof.** Let \( \mathfrak{Q} \) be the ideal which defines the non-Gorenstein locus of \( A \); we can observe that, as in Proposition 1.1, our conclusion and proof depend on height \( \mathfrak{Q} \). Precisely: \( \text{ht} \mathfrak{Q} = 0 \) implies both \( A \) does not contain any G-sequence of length \( = 0 \) (in fact in this case \( A \) is not \( G_0 \)) and \( A \) does not contain any G-sequence of positive length (namely if \( x = \{x_1, \ldots, x_r\} \) would be a G-sequence of length \( = r > 0 \), then for all \( i \in \{1, \ldots, r\} \) \( x_i \) would be a G-sequence which generates an unmixed ideal (cf. [S, Theorem 2.2]), so \( A/(x_i) \) and then \( A \) (by [R. F., Proposition 3]) would be \( S_1 \) and \( G_0 \) rings contradicting the fact that \( A \) is not \( G_0 \)).

\( \text{ht} \mathfrak{Q} > 0 \) implies \( \mathfrak{Q} \) is not a maximal G-sequence (we can use an argument like that of Proposition 1.1). Then let \( x \subseteq A \) be any maximal G-sequence of length \( = s > 0 \), if \( s < n - 1 \) for every \( \mathfrak{B} \in \text{Ass} (A/(x)), \mathfrak{B} \supseteq \mathfrak{Q}, (x) \) is an unmixed ideal by [S, Theorem 2.2], so there exists some element \( j \in \mathfrak{Q} \) which is a non-zero-divisor modulo \( (x) \). Considering the regular sequence \( \{x,j\} \), we can conclude, as in Proposition 1.1:

**Remark 1.** From [M3, Lemma 2.2] and [R. F., Proposition 3], we can deduce some information about the existence of G-sequences in a local ring \( A \) without having resort to the hypothesis that the Gorenstein locus of \( A \) is open; precisely we can say:

(i) In any local non-\( G_0 \) ring \( A \) which satisfies the “saturated chain condition on prime ideals”, there exist no G-sequences.

(ii) In any local non-\( G_0 \) ring \( A \) there exist no G-sequences (of any length) which generated unmixed ideals, but a priori we do not have any information about possible G-sequences which generated mixed ideals.

**Remark II.** From the proof of Corollary 1.3 we can deduce that in a local \( S_n \) ring \( A \) (\( K - \dim A > n > 1 \)) which is \( G_0 \) and has open Gorenstein locus, there exist G-sequences of length \( = \text{depth} A - 1 \) but a priori we cannot say if this must be the length of every maximal G-sequence of \( A \), so actually we do not know whether there may exist maximal G-sequences of different lengths in \( A \).

Recall explicitly the following notation introduced in [W.I.T.O., Definition 1.7].
Definition 1.4. A ring homomorphism \( \varphi: A \to B \) is Gorenstein if it is flat and has Gorenstein fibres.

Lemma 1.5. Let \( A \) be any ring with open Gorenstein locus. Then, for every Gorenstein homomorphism \( \varphi: A \to B \), the Gorenstein locus of \( B \) is open.

Proof. Let \( f: Y = \text{Spec } B \to \text{Spec } A = X \) be the induced morphism and let \( U \subset X \) be the Gorenstein locus of \( A \). For every \( \mathfrak{B} \in f^{-1}(U) \), \( B_{\mathfrak{B}} \) is Gorenstein; namely, putting \( \mathfrak{v} = f(\mathfrak{B}) \), clearly \( \mathfrak{v} \subset U \) so the local homomorphism \( \varphi: A_\mathfrak{v} \to B_\mathfrak{v} \) (induced by \( \varphi \)) is flat with \( A_\mathfrak{v} \) Gorenstein (since \( \mathfrak{v} \subset U \)) and \( B_{\mathfrak{B}/\mathfrak{v}B_{\mathfrak{B}}} \) Gorenstein (since it is a localization of the fibre of \( \varphi \) at \( \mathfrak{v} \)). On the other hand, for every \( \mathfrak{C} \subset Y - f^{-1}(U) \), \( B_{\mathfrak{C}} \) is not Gorenstein since putting \( \mathfrak{q} = \mathfrak{C} \cap \mathfrak{A} \) clearly \( \mathfrak{q} \notin U \), so \( A_{\mathfrak{q}} \) is not Gorenstein. Therefore the Gorenstein locus of \( B \) is precisely \( f^{-1}(U) \) which, by the hypothesis, is clearly open.

Proposition 1.6. Let \( (A, m) \) be any local C. M. ring with open Gorenstein locus. Then for every local Gorenstein homomorphism \( \varphi: A \to B \), we have

\[
\text{G - depth } B = \text{G - depth } A + \text{G - depth } B/mB.
\]

Proof. Observe that under the given hypotheses not only \( B \) is a (local) C. M. ring (cf. [D, Corollary 5.1]) but also its Gorenstein locus is open (cf. Lemma 1.5): therefore G-depth \( B \) is actually well defined (cf. Corollary 1.2). In addition, we notice that if \( A \) is Gorenstein, then there is nothing to prove, since in that case \( B \) is a (local) Gorenstein ring by [W.I.T.O., Theorem 1], so clearly

\[
\text{G - depth } B = K - \dim B = K - \dim A + K - \dim B/mB
\]

\[
= \text{G - depth } A + \text{G - depth } B/mB.
\]

Then, to show our contention, we only have to examine the case \( A \) is not Gorenstein which, according to Proposition 1.1, splits into G-depth \( A = -\infty \) and G-depth \( A = K - \dim A - 1 \) (automatically \( \geq 0 \)). If G-depth \( A = -\infty \), then we cannot have G-depth \( B \geq 0 \), since that would mean \( B \) is at least \( G_0 \) and hence also \( A \) would be at least \( G_0 \) (cf. [M1, Theorem 5.1]), contradicting G-depth \( A = -\infty \); then

\[
\text{G - depth } B = -\infty = -\infty + K - \dim B/mB
\]

\[
= \text{G - depth } A + \text{G - depth } B/mB.
\]

If \( 0 \leq \text{G - depth } A = K - \dim A - 1 \), then \( B \) is a \( G_0 \) ring (cf. [M1, Corollario 5.2]), that is G-depth \( B \geq 0 \). Moreover, since we are assuming \( A \) is not Gorenstein, we cannot have G-depth \( B = K - \dim B \) (cf. [W.I.T.O., Theorem 1]); thus again

\[
\text{G - depth } B = K - \dim B - 1 = K - \dim A - 1 + K - \dim B/mB
\]

\[
= \text{G - depth } A + \text{G - depth } B/mB.
\]

Remark III. Observe that if the local C. M. ring \((A, m)\) has the canonical module \( K_A \), then every local flat ring homomorphism \( \varphi: A \to B \) such that
$B/\mathfrak{m}B$ is Gorenstein (i.e. $K_B \simeq K_A \otimes_A B$ (cf. [H.K., Satz 6.14.])) is actually a Gorenstein homomorphism. Namely, using the same notations as in Lemma 1.5, for any $\mathfrak{p} \in X$, the fibre of $\varphi$ at $\mathfrak{p}$ is Gorenstein since (cf. [E.G.A., IV$_2$, Lemma 7.3.2]) for every $\mathfrak{p} \in f^{-1}(\mathfrak{p})$, $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is Gorenstein, being

$$(K_{A_\mathfrak{p}}) \otimes_{A_\mathfrak{p}} B_\mathfrak{p} = (K_A) \otimes_A A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{p} = K_A \otimes_A A_\mathfrak{p} = K_A \otimes_A B_\mathfrak{p} = K_A \otimes_A B_\mathfrak{p}$$

(cf. [R, (3) Theorem] and [H.K., Korollar 5.25]). Notice that the hypothesis "$B/\mathfrak{m}B$ is Gorenstein" cannot be avoided; namely, we can easily see that Proposition 1.6 does not hold for a local flat homomorphism $\varphi: A \to B$ of complete, equidimensional local rings such that $A$ is Gorenstein and $B$ is $G_0$ but not Gorenstein (e.g., fix a field $k$, take

$$A = k[[U^N]] \quad (N = 1, 2),$$
$$B = k[[x,y,z,t]]/(t^2 - x^3, x^2 - y, y^2 - xz, yz - xt),$$

and $\varphi$ the inclusion map (which is clearly flat and local); what we get is $G$ - depth $A = 1$, $G$ - depth $B = 0$ ($B$ is a 1-dimensional complete integral domain which is not Gorenstein (cf. [K, Theorem]), and $G$ - depth $B/\mathfrak{m}B = -\infty$).

Finally we are going to list explicitly some examples of local flat ring homomorphisms for which Proposition 1.6 holds.

1.7. Let $(A, \mathfrak{m})$ be any local CM ring with residue field $k$ such that $K_A$ exists. Then:

(i) If $x$ is an indeterminate, for every maximal ideal $\mathfrak{m} \subset A[x]$ such that $\mathfrak{m} \cap A = \mathfrak{m}$, we have $G$ - depth $A[x]_{\mathfrak{m}} = G$ - depth $A + 1$ (in fact the fibre $k \otimes_A A[x]_{\mathfrak{m}}$ is isomorphic to the 1-dimensional Gorenstein ring $k[x]$ localized at the maximal ideal $\mathfrak{m} = \mathfrak{m}k[x]$ (cf. [G.S., Example 12.1])).

(ii) If $F$ is a finite abelian group, for every maximal ideal $\mathfrak{m} \subset A[F]$, we have $G$ - depth $A[F]_{\mathfrak{m}} = G$ - depth $A$ (in fact $k \otimes_A A[F]_{\mathfrak{m}}$ is isomorphic to the 0-dimensional Gorenstein ring $k[F]$ localized at the maximal ideal $\mathfrak{m} = \mathfrak{m}k[F]$ (cf. [P, Corollaire 2])).

(iii) If $\hat{A}$ is the henselization of $A$ with respect to $\mathfrak{m}$, we have $G$ - depth $\hat{A} = G$ - depth $A$ (in fact $k \otimes_A \hat{A} \simeq \hat{A}/\mathfrak{m} \hat{A} \simeq k$ (cf. [E.G.A., IV$_4$, Theorem 18.6.6])).

(iv) If $\hat{A}$ is the $m$-adic completion of $A$, we have $G$ - depth $\hat{A} = G$ - depth $A$ (in fact $k \otimes_A \hat{A} \simeq \hat{A}/\mathfrak{m} \hat{A} \simeq k$ (cf. [D, §6.A5])).

(v) If $A[[x]]$ is the formal power series ring (in one indeterminate) over $A$, we have $G$ - depth $A[[x]] = G$ - depth $A + 1$ (in fact the fibres of $A \to A[[x]]$ are canonically isomorphic to the fibres of $B \to B[[x]]$ (where $B$ is a local Gorenstein ring such that $A = B/b$ (cf. [R, (3) Theorem])) at the prime ideals of $B$ containing $b$, and $B \to B[[x]]$ is a local flat ring homomorphism whose fibre at the closed point of Spec $B$ is equal to $k \otimes_B B[[x]]$ which is clearly a (local) 1-dimensional Gorenstein ring (cf. [G.S., Theorem 9.8] and [W.I.T.O., Theorem 2])).
References


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