THE Q-TOPOLOGY, WHYBURN TYPE FILTERS 
AND THE CLUSTER SET MAP

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ABSTRACT. We use nonstandard topology and the Q-topology to characterize normal, almost-normal, regular, almost-regular, semiregular spaces. The cluster [resp. \( \theta \)-cluster] set relation is used to characterize regular, almost-regular [resp. strongly-regular] spaces. The Whyburn [resp. Dickman] filter bases are characterized and it is shown that the cluster [resp. \( \theta \)-cluster] set relation restricted to the domain of the Whyburn [resp. Dickman] filter bases is an essentially continuous [resp. strongly \( \theta \)-continuous] map iff the space is Hausdorff [resp. Urysohn].

1. Introduction. This paper has three major purposes. First, we investigate the Q-topology on an enlargement \( *X \) of a topological space \( X \) as introduced by Robinson [9] and show, among other results, that the Q-closure of a point or set monad is the \( \theta \)-monad [6]. Moreover, using the Q-topology and the point or set monad, we characterize regular, semiregular, almost-regular [10], normal and almost-normal [11] spaces by means of a collection of highly analogous statements.

Fuller [4] defines a topology on the set of all clustering filters on \( X \) and using the lower semifinite topology shows that the cluster set map is continuous iff \( X \) is locally compact. Employing a different topology on the set of all converging filters, Wyler [13] shows that the convergence of a filter on a Hausdorff space \( X \) is a continuous map iff \( X \) is regular. We use that standard part [resp. \( \theta \)-standard part] relation, which can be considered the cluster [resp. \( \theta \)-cluster] set map, and show that, from the nonstandard viewpoint, regular [resp. almost-regular, strongly-regular] spaces are characterizable by similar statements involving the inverse of this relation. Further, by considering the near-standard [resp. \( \theta \)-near-standard] points and employing the induced Q-topology, we show that the cluster [resp. \( \theta \)-cluster] set relation is a continuous map iff \( X \) is Hausdorff [resp. Urysohn].

In [12], Whyburn introduces the concept of a filter base being directed toward \( A \subset X \) and uses this concept to characterize perfect (not necessarily continuous) maps. Dickman [2], [3] modifies Whyburn's definition and introduces the concept of a filter base almost-converging to \( A \subset X \). Among our final results, we show that a filter base is directed toward [resp. almost-converges to] \( A \subset X \) iff its nucleus satisfies a nonstandard condition analo-
gous to the criterion for compactness [resp. quasi-\(H\)-closedness].

Throughout this paper, we let \( \mathcal{M} = (\mathcal{U}, \in, \text{pr}, \text{ap}) \) be the standard set-theoretic structure constructed by Machover and Hirschfeld [8] and, as usual, assume that all standard objects are elements of \( \mathcal{U} \). Even though some of the results only require \( \ast\mathcal{M} = (\ast\mathcal{U}, \ast\in, \ast\text{pr}, \ast\text{ap}) \) to be an enlargement, it is convenient to assume that the extension \( \ast\mathcal{M} \) is \( \kappa \)-saturated, where \( \kappa \) is any cardinal larger than the cardinality of \( \mathcal{M} \). In the usual manner [7], [8], [9] we let \( \mathcal{E} \) be a first order language with equality and the usual assortment of abbreviations which formally describes \( \mathcal{M} \). Also we do not distinguish between the formal constant, relation and operator symbols in \( \mathcal{E} \) and the corresponding objects in \( \mathcal{M} \). We assume that the reader is familiar with the concepts and methods associated with nonstandard topology [7], [8], [9]. We use much of the notation found in [8].

2. The \(Q\)-topology. For a topological space \((X, \tau)\), the \(Q\)-topology on \(\ast X\), denoted by \(\mathcal{Q}\), is the topology generated by \((\ast A | A \in \ast \tau)\) as a base. Recall that if \(A \in \ast \mathcal{U}\), then \(\ast A = \{ p | p \in \ast \mathcal{U} \} \land \{ p \in \ast A \}\). If \(A \in \ast \tau\), then \(\ast A\) is said to be \(\ast\)-open. If \(B \in \mathcal{Q}\), then \(B\) is said to be \(Q\)-open, etc. We let \(\mu(p)\) and \(\mu(A)\) be the point and set monad [9] and define

\[
\mu_\alpha(p) = \bigcap\{ (\text{int}_X G) | p \in G \in \tau\},
\]

\[
\mu_\theta(A) = \bigcap\{ (\text{cl}_X G) | A \subset G \in \tau\},
\]

\[
\mu_\alpha(p) = \bigcap\{ (\text{int}_X G) | p \in G \in \tau\},
\]

\[
\mu_\theta(A) = \bigcap\{ (\text{cl}_X G) | A \subset G \in \tau\}.
\]

to be the \(\alpha\) and \(\theta\) point and set monads respectively.

For many properties of the \(Q\)-topology not mentioned in this paper, we refer the reader to [1], [9]. In particular, Button [1] has shown that the \(Q\)-topology preserves much of the structure of \(\tau\) and, indeed, \((\ast X, \mathcal{Q})\) is discrete iff \((X, \tau)\) is discrete.

**Theorem 2.1.** If nonempty \(\mathcal{G} \subset \tau\), then \(\text{Nuc}\mathcal{G}\) is \(Q\)-open.

**Proof.** If \(\mathcal{G}\) does not have the finite intersection property, then \(\text{Nuc}\mathcal{G} = \emptyset\). Assume that \(\mathcal{G}\) has the finite intersection property and let \(\mathcal{G}\) be the open filter generated by \(\mathcal{G}\). Luxemburg's Theorem 2.1.6 [7] holds for any filter on any meet-semilattice of sets [5]. Hence \(\text{Nuc}\mathcal{G} = \bigcup\{ (\ast E | E \in \mathcal{G}) \land (\ast E \subset \text{Nuc}\mathcal{G})\}\).

**Corollary 2.1.1.** For each \(p \in X\) and \(A \subset X\), the monads \(\mu(p), \mu(A), \mu_\alpha(p), \mu_\theta(A)\) are \(Q\)-open.

**Remark.** In [1], Button obtains 2.1 by using a considerably more elaborate technique.

Clearly, if \(\mathcal{G}\) is an open filter on \(X\), then every infinitesimal \(\ast\)-element in \(\mathcal{G}\) is \(\ast\)-open. Indeed, we have a converse to this assertion.

**Theorem 2.2.** Let \(\mathcal{F}\) be a filter base on \(X\). If each infinitesimal \(\ast\)-element in \(\mathcal{F}\) is \(\ast\)-open, then \(\text{Nuc}\mathcal{F} = \text{Nuc}\mathcal{G}\), where \(\mathcal{G} = \{ G | G \in \tau \land (G \in \mathcal{F})\}\).
Proof. Since $\mathcal{F}$ is a filter base, then there exists an infinitesimal $*\text{element in } \mathcal{F}$. Thus, it follows by transfer that $\mathcal{F} = \{ G \mid G \in *\mathcal{F} \} \neq \emptyset$. Clearly, $\text{Nuc}_\mathcal{F} \subset \text{Nuc}_G$. Now let $F \in \mathcal{F}$ and $G = \{ E \mid E \in *\tau \} \land \{ E \in \mathcal{F} \} \land \{ *E \subset *F \}$. Using saturation and Luxemburg's Theorem 2.7.3(c) [7], which also holds for filter bases, we have that there exists an open $G \in \mathcal{F}$ such that $G \subset F$. Consequently, $\text{Nuc}_G \subset \text{Nuc}_\mathcal{F}$ and the result follows.

Clearly, for $A \subset X$, $*(\text{cl}_X A)$ is $*$-closed. Hence $\mu_\theta(p)$ and $\mu_\theta(A)$ are $Q$-closed. Of course, $\text{ns}(X) = \bigcup \{ (\mu(p) \mid p \in X) \}$ is $Q$-open.

Theorem 2.3. For each $p \in X$ [resp. $A \subset X$], the monad $\mu_\theta(p) = \text{cl}_X (\mu(p))$ [resp. $\mu_\theta(A) = \text{cl}_X (\mu(A))$].

Proof. We only show the first assertion, the second being similar. Let $p \in X$. Since $\mu(p) \subset \mu_\theta(p)$, then $\text{cl}_X (\mu(p)) \subset \mu_\theta(p)$. Assume that there exists $q \in \mu_\theta(p)$ and $q \notin \text{cl}_X (\mu(p))$. Now there exists $E \in *\tau$ such that $q \in *E$ and $*E \cap \mu(p) = \emptyset$. Saturation implies that there exists $G \in \tau$ such that $p \in G$ and $*E \cap *G = \emptyset$. Hence $*E \cap *(\text{cl}_X G) = \emptyset$ by transfer. However, $q \in \mu_\theta(p)$ implies $*E \cap *(\text{cl}_X G) \neq \emptyset$ and the result follows.

Since $X$ is regular [resp. almost-regular [10]] iff $\mu(p) = \mu_\theta(p)$ [resp. $\mu_\theta(p) = \mu_\theta(p)$] for each $p \in X$ [6], then it follows that a space $X$ is regular [resp. almost-regular] iff $\mu(p)$ [resp. $\mu_\theta(p)$] is $Q$-closed for each $p \in X$. Also, it is easy to show that a space $X$ is normal [resp. almost-normal [11]] iff $\mu(A) = \mu_\theta(A)$ [resp. $\mu_\theta(A) = \mu_\theta(A)$] for each closed $A \subset X$. Hence a space $X$ is normal [resp. almost-normal] iff $\mu(A)$ [resp. $\mu_\theta(A)$] is $Q$-closed for each closed $A \subset X$.

Remark. Button [1], using a different technique, also gives the $Q$-open and $Q$-closed characterizations for regular and normal spaces.

In [6], we give some nonstandard characterizations for semiregular spaces. Using the $Q$-topology, we obtain another characterization. Let $\tau_5$ be the topology generated by the set of all regular-open subsets in $X$ and $\mathcal{F}_5$ its associated $Q$-topology.

Theorem 2.4. A space $(X, \tau)$ is semiregular iff $\mu(p) \in \mathcal{F}_5$ for each $p \in X$.

Proof. For the necessity, let $(X, \tau)$ be semiregular. Then $\mathcal{F}_5 = \mathcal{F}$. Thus applying 2.1.1, we have that $\mu(p) \in \mathcal{F}_5$ for each $p \in X$.

For the sufficiency, let $\mu(p) \in \mathcal{F}_5$. Since $*\tau_5$ is a base for $\mathcal{F}_5$ and $p \in \mu(p)$, then it follows that there exists $E \in *\tau_5$ such that $p \in *E \subset \mu(p) \subset \mu_\theta(p)$. Let $G$ be any open set such that $p \in G$. Then the sentence in $\mathcal{M}$,

$$\exists x [x \in \tau_5] \land [p \in X] \land [x \subset G],$$

holds in $\mathcal{M}$ by transfer. Consequently, since we are dealing with filter bases, we have that $\mu_\theta(p) \subset \mu(p)$. Thus $\mu(p) = \mu_\theta(p)$. This implies that $(X, \tau)$ is semiregular [6].

3. The cluster set map. As is well known if $\mathcal{F}$ is a filter base on $X$, then $\text{St} [\text{Nuc}_\mathcal{F}]$ is the cluster set for $\mathcal{F}$, where for $W \subset *X$, $\text{St} [W] = \{ p \mid \text{p} \in X \land \{ \text{p} \land W \neq \emptyset \} \}$. Recall that a set $W \subset *X$ is nuclear if there exists $\mathcal{F} \subset \mathcal{P}(X)$ such that $W = \text{Nuc}_\mathcal{F}$. Hence "St" restricted to $\text{ns}(X)$ is essentially the cluster set map for filter bases on $X$. Of course, in this case "St" may be considered a map from $\text{ns}(X)$ into $X$ if $X$ is Hausdorff. A space $(X, \tau)$ is
called strongly-regular if for closed \( F \subseteq X \) and \( p \in X - F \) there exist \( G, H \in \tau \) such that \( p \in G, F \subseteq H \) and \( \text{cl}_X G \cap \text{cl}_X H = \emptyset \). Observe that completely regular implies strongly-regular implies regular.

**Definition 3.1.** For each \( W \subseteq X \), let \( \text{St}_\theta[W] = \{ p | [p \in X] \land [\mu_\theta(p) \cap W \neq \emptyset] \} \) and \( \text{ns}_\theta(X) = \bigcup \{ \mu_\theta(p) | p \in X \} \). Notice that if \( \mathcal{F} \) is a filter base, then \( \text{St}_\theta[\text{Nuc}\mathcal{F}] \) is the set of all \( \theta \)-cluster points [3] for \( \mathcal{F} \). Also, "\( \text{St}_\theta \)" is a map from \( \text{ns}_\theta(X) \) into \( X \) iff \( X \) is Urysohn [6] (i.e. for distinct \( p, q \in X \) there exist neighborhoods \( N_p, N_q \) such that \( \text{cl}_X N_p \cap \text{cl}_X N_q = \emptyset \)).

**Theorem 3.1.** Let \( (X, \tau) \) be Hausdorff and \( \text{St} : \text{ns}(X) \to X \). Then:

(i) \( X \) is regular iff \( \text{St}_\tau^{-1}[F] = \mu(F) \cap \text{ns}(X) \) for each closed \( F \subseteq X \).

(ii) \( X \) is almost-regular iff \( \text{St}_\tau^{-1}[F] = \mu_a(F) \cap \text{ns}(X) \) for each regular-closed \( F \subseteq X \).

**Proof.** (i) For the necessity, let closed \( F \subseteq X \) and \( q \in \text{St}_\tau^{-1}[F] \). Then \( \text{St}(q) = p \) implies that \( q \in \mu(p) \) and \( \mu(p) \cap \text{cl}_X F \neq \emptyset \). Hence \( p \in F \). Thus \( \mu(p) \subseteq *G \) for each open \( G \supseteq F \). Consequently, \( q \in \mu(F) \cap \text{ns}(X) \) implies that \( \text{St}_\tau^{-1}[F] \subseteq \mu(F) \cap \text{ns}(X) \). Now assume that \( X \) is regular and \( q \in \mu(F) \cap \text{ns}(X) \). Then \( q \in \mu(p) \) for some \( p \in X \). Assume that \( p \notin F \). Then there exist disjoint \( G, H \in \tau \) such that \( p \in G \) and \( F \subseteq H \). Thus \( \mu(p) \cap *H = \emptyset \). Consequently, \( \text{St}_\tau[\mu(p)] = p \in F \) and the necessity follows.

For the sufficiency, let closed \( F \subseteq X \) and \( p \notin F \). Then \( \text{St}_\tau^{-1}(p) = \mu(p) \subseteq \text{ns}(X) \) and \( \text{St}_\tau^{-1}[F] = \mu(F) \cap \text{ns}(X) \). Observe that \( \mu(p) \cap *F = \emptyset \). Hence

\[
\emptyset = \text{St}_\tau^{-1}[F \cap \{ p \}] = \text{St}_\tau^{-1}[F] \cap \text{St}_\tau^{-1}(p) = \mu(F) \cap \text{ns}(X) \cap \mu(p) = \mu(F) \cap \mu(p).
\]

Thus there exist disjoint \( G, H \in \tau \) such that \( p \in G \) and \( F \subseteq H \).

(ii) Observe that if \( F \subseteq X \) is regular-closed in \( X \), then

\[
\text{St}_\tau^{-1}[F] \subseteq \mu(F) \cap \text{ns}(X) \subseteq \mu_a(F) \cap \text{ns}(X).
\]

The result follows in the same manner as in (i) since the operator "\( \text{int}_X \text{cl}_X \)" preserves disjointness for open sets.

Clearly, a strongly-regular \( T_1 \) space is Urysohn. Of course, since a strongly-regular space is regular, then in a strongly-regular space \( X, F \subseteq X \) is closed iff \( \text{St}_\theta[*F] = F \). The following result is obtained in the same manner as is Theorem 3.1.

**Theorem 3.2.** Let \( X \) be Urysohn. Then \( X \) is strongly-regular iff \( \text{St}_\theta^{-1}[F] = \mu_\theta(F) \cap \text{ns}_\theta(X) \) for each closed \( F \subseteq X \).

4. Whyburn and Dickman filter bases. In [12], Whyburn says that a filter base \( \mathcal{F} \) on \( X \) is directed toward \( A \subseteq X \) if every filter base \( \mathcal{G} \) stronger than \( \mathcal{F} \) has a cluster point in \( A \). Dickman [3] modifies Whyburn's definition and says that a filter base \( \mathcal{F} \) on \( X \) is almost-convergent to \( A \subseteq X \) if every filter base \( \mathcal{G} \) stronger than \( \mathcal{F} \) has an almost-cluster point in \( A \) (i.e. \( \text{St}_\theta[\text{Nuc}\mathcal{G}] \cap A \neq \emptyset \)). We call a filter base \( \mathcal{F} \) a Whyburn [resp. Dickman] filter base if \( \mathcal{F} \) is directed toward [resp. almost-convergent to] some \( A \subseteq X \).
Definition 4.1. A set $W \subset \mathcal{X}$ is $A$-compact [resp. $\theta A$-compact] for $A \subset X$ if $W \subset \bigcup \{\mu(p) | p \in A\}$ [resp. $\{\mu_\theta(p) | p \in A\}$].

Theorem 4.1. Let $\mathcal{F}$ be a filter base on $X$. Then the following statements are equivalent.

(i) For each open cover $C$ of $A$, we have that $\text{Nuc}\mathcal{F} \subset \bigcup \{*G|G \in C\}$ [resp. $\{*(\text{cl}_X G)|G \in C\}$].

(ii) $\text{Nuc}\mathcal{F}$ is $A$-compact [resp. $\theta A$-compact].

(iii) For each open cover $C$ of $A$ there exists a finite subcover $\mathbb{G}$, such that $\text{Nuc}\mathcal{F} \subset \bigcup \{*D|D \in \mathbb{G}\}$ [resp. $\{*(\text{cl}_X D)|D \in \mathbb{G}\}$].

(iv) For each open cover $C$ of $A$ there exists a finite subcover $\mathbb{G}$ and an $F \in \mathcal{F}$, such that $F \subset \bigcup \{D|D \in \mathbb{G}\}$ [resp. $\{\text{cl}_X D|D \in \mathbb{G}\}$].

Proof. We only prove the first conclusions since the second follow in a similar manner.

(i) $\rightarrow$ (ii). Assume that $q \in \text{Nuc}\mathcal{F}$ and $q \notin \bigcup \{\mu(p) | p \in A\}$. Then for each $p \in A$ there exists some open neighborhood $G$ such that $q \notin *G$. Thus $C = \{G[G \in \tau] \land [q \notin *G]\}$ is an open cover of $A$ such that $\text{Nuc}\mathcal{F} \notin \bigcup \{*G|G \in C\}$.

(ii) $\rightarrow$ (iii). Assume that there exists some open cover $C$ of $A$ such that for no finite $\mathbb{G} \subset C$ do we have that $\text{Nuc}\mathcal{F} \subset \bigcup \{*D|D \in \mathbb{G}\}$.

(iii) $\rightarrow$ (iv). Simply let $E$ be the infinitesimal element which is contained in $\text{Nuc}\mathcal{F}$.

(iv) $\rightarrow$ (i) is obvious.

Corollary 4.1.1. A filter base $\mathcal{F}$ on $X$ is Whyburn [resp. Dickman] iff $\text{Nuc}\mathcal{F} \subset \text{ns} (*X)$ [resp. $\text{Nuc}\mathcal{F} \subset \text{ns}_\theta (*X)$].

Corollary 4.1.2. A filter base $\mathcal{F}$ on $X$ is directed toward [resp. almost-converges to] $A \subset X$ iff $\text{Nuc}\mathcal{F}$ is $A$-compact [resp. $\theta A$-compact].

Remark. The reader may wish to compare Theorem 4.1 with the known
results that a set $A \subseteq X$ is compact [resp. quasi-$H$-closed relative to $X$] iff $^*A$ is $A$-compact [9] [resp. $\theta A$-compact [6]].

Recall that a map $f: X \to Y$ is strongly $\theta$-continuous at $p \in X$ if for every open neighborhood $N$ of $f(p)$ there exists some open neighborhood $G$ of $p$ such that $f[\text{cl}_X G] \subseteq N$. Since in the $Q$-topology $\mu(p)$ is open and $\text{cl}_X (\mu(p)) = \mu_\theta(p)$ for each $p \in X$, then the next result follows easily and compares nicely with the results of Fuller [4] and Wyler [13].

**Theorem 4.2.** Let $\text{ns} (^*X)$ [resp. $\text{ns}_\theta (^*X)$] carry the topology induced by the $Q$-topology on $^*X$. Then $\text{St}: \text{ns} (^*X) \to X$ [resp. $\text{St}_\theta: \text{ns}_\theta (^*X) \to X$] is a continuous [resp. strongly $\theta$-continuous] map iff $X$ is Hausdorff [resp. Urysohn].

**References**

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