A NOTE ON WALSH-FOURIER SERIES

WO-SANG YOUNG

Abstract. It is shown that the double sequence \( \{ \lambda_{mn} \} \) with \( \lambda_{mn} = 1 \) if \( n < m \) and 0 otherwise is an \( L^p \) multiplier for the Walsh system in two dimensions only if \( p = 2 \). This result is then used to show that the one-dimensional trigonometric system and the Walsh system are nonequivalent bases of the Banach space \( L^p[0, 1] \), and hence have different \( L^p \) multipliers, \( 1 < p < \infty, p \neq 2 \).

1. Let \( \{ \lambda_{mn} \}_{-\infty < m,n < \infty} \) be the double sequence defined by \( \lambda_{mn} = 1 \) if \( n < m \) and 0 otherwise. For

\[
\sum_{m,n=-\infty}^{\infty} a_{mn} e^{2\pi i(mx + ny)} \in L^p([0,1] \times [0,1]),
\]

let

\[
T_1 f \sim \sum_{m,n=-\infty}^{\infty} a_{mn} \lambda_{mn} e^{2\pi i(mx + ny)}.
\]

It is well known that \( T_1 \) is bounded on \( L^p([0,1] \times [0,1]), 1 < p < \infty \). This is a consequence of the one-dimensional result of M. Riesz for the conjugate function [6, I, p. 253]. The \( L^p \) boundedness of \( T_1 \) was used, for example, in C. Fefferman's proof of the almost everywhere convergence of double Fourier series [1].

We now turn our attention to the Walsh system \( \{ w_n \} \). For

\[
\sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y) \in L^p([0,1] \times [0,1]),
\]

consider the corresponding operator \( T_2 \) defined by

\[
T_2 f \sim \sum_{m,n=0}^{\infty} a_{mn} \lambda_{mn} w_m(x) w_n(y).
\]

Because of the great similarity between the Walsh system and the trigonometric system, one would expect \( T_2 \) to be bounded on \( L^p([0,1] \times [0,1]), 1 < p < \infty \). However, this is not the case. In §2 we will show that \( T_2 \) is not bounded on \( L^p \) except for \( p = 2 \). This result is then used in §3 to give a negative answer to a question of P. Enflo: Are the trigonometric system and the Walsh system equivalent bases of the Banach space \( L^p[0, 1], 1 < p < \infty, p \neq 2 \)? Finally in §4 we will deduce from the nonequivalence that the
one-dimensional Walsh system and trigonometric system have different $L^p$ multipliers, $1 < p < \infty$, $p \neq 2$.

2. Let $\{r_n\}_{n \geq 0}$ denote the Rademacher functions and $\{w_n\}_{n \geq 0}$ the Walsh functions defined on $I = [0, 1]$. For any two real numbers with dyadic expansions $a = \sum_{j=-\infty}^{\infty} a_j 2^j$, $b = \sum_{j=-\infty}^{\infty} \beta_j 2^j$, $a_j, \beta_j = 0$ or $1$, let $a + b = \sum_{j=-\infty}^{\infty} |a_j - \beta_j| 2^j$. It is understood that we use the finite representation in the case of a dyadic rational. Basic properties of the Walsh functions can be found in [2].

**Theorem 1.** $T_2$ is bounded on $L^p(I^2)$ if and only if $p = 2$.

**Proof.** The case $p = 2$ is trivial by Parseval's formula. It is sufficient to show that $T_2$ is not bounded on $L^p$ for $p < 2$, for then the theorem will follow by a duality argument.

Instead of dealing with $T_2$ we consider an equivalent operator $T'_2$ defined on $L^p(I^2)$ as follows. Let $A = \{(m, n): 0 < n < m + n, m = 0, 1, \ldots\}$. For $f \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y)$, let $T'_2 f \sim \sum_{(m,n) \in A} a_{mn} w_m(x) w_n(y)$. Suppose $g$ is the function on $I^2$ defined by $g(x, y) = f(x, x + y)$. Since $w_n(x + y) = w_n(x) w_n(y)$ and $w_{m+n} = w_m w_n$, we have $T'_2 g(x, y) = T_2 f(x, x + y)$. Therefore the boundedness of $T_2$ is equivalent to that of $T'_2$.

We will define a sequence of functions $\{f_k\}$ on $I^2$ and show that

$$\|T'_2 f_k\|_p / \|f_k\|_p \to \infty \quad \text{as } k \to \infty.$$ 

Let

$$f_k = \sum_{k=1}^{k} f_{kl},$$

where

$$f_{kl}(x, y) = 2^{-k} r_{l-1}(x) \prod_{j=0}^{k-1} (1 + r_j(y)).$$

We first note that

$$\|f_{kl}\|_p^p = \int \left| 2^{-k} \prod_{j=0}^{k-1} (1 + r_j(y)) \right|^p dy = 2^{-kp} \int_{[0,2^{-k}]} 2^{kp} dy = 2^{-k}.$$ (1)

Moreover, since $\int |\sum_{l=0}^{k} r_l(x)|^p dx \leq C_p k^{p/2}$ by Khintchin's inequality [6, 1, p. 213], we have

$$\|f_k\|_p^p = \int \left| \sum_{l=0}^{k} r_l(x) \right|^p dx \int \left| 2^{-k} \prod_{j=0}^{k-1} (1 + r_j(y)) \right|^p dy \leq C_p k^{p/2} 2^{-k},$$ (2)

where $C_p$ denotes a constant depending only on $p$.

On the other hand, for $1 < l \leq k$,

$$T'_2 f_{kl}(x, y) = 2^{-k} \sum_{n < m + n, m} \int r_{l-1}(s) w_m(x) \, ds \cdot \int \prod_{j=0}^{k-1} (1 + r_j(t)) w_n(t) \, dt w_m(x) w_n(y).$$
Now
\[ \int r_{l-1}(s)w_m(s) \, dx = 1 \quad \text{if } m = 2^{l-1} \]
and 0 otherwise. Also
\[ \int \prod_{j=0}^{k-1} (1 + r_j(t))w_n(t) \, dt = 1 \quad \text{if } n < 2^k \]
\[ = 0 \quad \text{if } n > 2^k. \]
Therefore
(3) \[ T_2f_{kl}(x,y) = 2^{-k} \sum_{n < 2^{l-1} + n: n < 2^k} w_n(y)w_{2^{l-1}}(x). \]
Let \( n = \sum_{j=0}^{\infty} \epsilon_j 2^j \) with \( \epsilon_j = 0 \) or 1. We observe that \( n < 2^{l-1} + n \) if and only if \( \epsilon_{l-1} = 0. \) Therefore
\[ \sum_{n < 2^{l-1} + n: n < 2^k} w_n(y) = \prod_{0 < j < k;j \neq l-1} (1 + r_j(y)) \]
(4) \[ = \frac{1}{2} \left[ \prod_{0 < j < k} (1 + r_j(y)) + \prod_{0 < j < k} (1 + r_j(y + 2^{-l})) \right]. \]
From (3) and (4), we have
\[ T_2f_{kl}(x,y) = \frac{1}{2} \left[ f_{kl}(x,y) + f_{kl}(x,y + 2^{-l}) \right], \]
and hence
\[ T_2f_k(x,y) = \frac{1}{2} f_k(x,y) + \frac{1}{2} \sum_{l=1}^{k} f_{kl}(x,y + 2^{-l}). \]
Since \( f_k(x,y) \) and \( f_{kl}(x,y + 2^{-l}), \quad l = 1, \ldots, k, \) have mutually disjoint supports, it follows from (1) that
(5) \[ \|T_2f_k\|_p^p \geq 2^{-p} \sum_{l=1}^{k} \|f_{kl}\|_p^p = 2^{-p} k 2^{-k}. \]
Combining (2) and (5), we obtain, for \( p < 2, \)
\[ \|T_2f_k\|_p^p / \|f_k\|_p^p \geq 2^{-1} C_p^{-1} k^{(1/p - 1/2)} \to \infty \quad \text{as } k \to \infty. \]
This completes the proof of Theorem 1.

3. It is known that \( \{\cos \pi nx\} \) and \( \{w_n\} \) are bases of the Banach space \( L^p(I), \quad 1 < p < \infty. \) (See [6, 1, p. 266] and [4].) We say that the sequences \( \{u_n\}, \{v_n\} \) of a Banach space are equivalent if for every sequence of numbers \( \{a_n\}, \sum_{n=0}^{\infty} a_n u_n \) converges if and only if \( \sum_{n=0}^{\infty} a_n v_n \) converges. R. Askey, S. Wainger and J. E. Gilbert showed that \( \{\cos \pi nx\} \) and certain classical orthonormal sequences are equivalent in \( L^p(I), \quad 1 < p < \infty. \) (See [3].) We have the following

THEOREM 2. Let \( 1 < p < \infty. \) \( \{\cos \pi nx\} \) and \( \{w_n\} \) are equivalent bases of \( L^p(I) \) if and only if \( p = 2. \)

PROOF. Again the case \( p = 2 \) is trivial by Parseval’s formula. Suppose they
were equivalent in $L^p(I)$, $p \neq 2$. From this it would follow that \( \{e^{2\pi i n x}\}_{n \geq 0} \) and \( \{w_n\} \) are also equivalent in $L^p(I)$. (See [6, I, p. 253].) By the Banach-Steinhaus theorem, there exist constants $C_p$, $C_p' > 0$ such that for any sequence of numbers \( \{a_n\} \),

$$\left(6\right) \quad C_p^{-1} \left\| \sum_{n=0}^{N} a_n e^{2\pi i n x} \right\|_p \leq \left\| \sum_{n=0}^{N} a_n w_n \right\|_p \leq C_p' \left\| \sum_{n=0}^{N} a_n e^{2\pi i n x} \right\|_p , \quad N > 0. $$

(See [5, p. 70].) Let \( \{a_{mn}\} \) be any double sequence of numbers. Applying (6) first to the \( x \)-variable and then to the \( y \)-variable, we obtain

$$\left(7\right) \quad C_p^{-2} \left\| \sum_{m,n=0}^{N} a_{mn} e^{2\pi i(mx + ny)} \right\|_p \leq \left\| \sum_{m,n=0}^{N} a_{mn} w_m(x) w_n(y) \right\|_p \leq C_p'^2 \left\| \sum_{m,n=0}^{N} a_{mn} e^{2\pi i(mx + ny)} \right\|_p , \quad N > 0. $$

Now, it follows from the corresponding one-dimensional result that for any function in $L^p(I^2)$, the square partial sums of both its trigonometric Fourier series and Walsh-Fourier series converge in $L^p(I^2)$. Therefore (7) implies the following: for any sequence of numbers \( \{a_{mn}\} \), \( f \sim \sum_{m,n=0}^{\infty} a_{mn} e^{2\pi i(mx + ny)} \) for some \( f \in L^p(I^2) \) if and only if \( g \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y) \) for some \( g \in L^p(I^2) \).

Suppose \( g \in L^p(I^2) \) with \( g \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y) \). Then \( f \sim \sum_{m,n=0}^{\infty} a_{mn} e^{2\pi i(mx + ny)} \in L^p(I^2) \). Hence

$$T_1 f \sim \sum_{m,n=0}^{\infty} a_{mn} \lambda_{mn} e^{2\pi i(mx + ny)} \in L^p(I^2),$$

which implies

$$T_2 g \sim \sum_{m,n=0}^{\infty} a_{mn} \lambda_{mn} w_m(x) w_n(y) \in L^p(I^2).$$

Therefore $T_2$ maps $L^p(I^2)$ into $L^p(I^2)$. By the closed graph theorem, $T_2$ is bounded on $L^p(I^2)$, contradicting Theorem 1. This proves Theorem 2.

4. Let \( \{u_n\} \) be one of the sequences \( \{w_n\} \), \( \{\cos \pi n x\} \) or \( \{e^{2\pi i n x}\}_{n \geq 0} \). \( M(L^p, \{u_n\}) \) denotes the collection of all sequences \( \{\lambda_n\} \) such that \( f \sim \sum_{n=0}^{\infty} a_n u_n \in L^p(I) \) implies \( g \sim \sum_{n=0}^{\infty} \lambda_n a_n u_n \in L^p(I) \). We will deduce from Theorem 2 the following

**Theorem 3.** \( M(L^p, \{\cos \pi n x\}) \neq M(L^p, \{w_n\}) \), $1 < p < \infty$, $p \neq 2$.

We note that in general two nonequivalent bases of $L^p(I)$ may have the same multipliers. See, for example, [5, p. 484 and p. 546].

**Proof.** Suppose they were equal. Since

$$M(L^p, \{\cos \pi n x\}) = M(L^p, \{e^{2\pi i n x}\}_{n \geq 0}),$$

we have \( M(L^p, \{w_n\}) = M(L^p, \{e^{2\pi i n x}\}_{n \geq 0}) \). Let \( \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \in L^p \). For every \( t \in [0, 1] \),

$$\left\| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} e^{2\pi i n t} \right\|_p = \left\| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \right\|_p.$$
so \( \{e^{2\pi n i x}\} \in M(L^p, \{e^{2\pi n i x}\}) \), and hence belongs to \( M(L^p, \{w_n\}) \). We assert that, moreover, there is a constant \( C_p \), depending only on \( p \), such that for every \( \sum_{n=0}^{\infty} a_n w_n \in L^p \),

\[
(8) \quad \int \left| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} w_n(x) \right|^p dx \leq C_p ^p \int \left| \sum_{n=0}^{\infty} a_n w_n(x) \right|^p dx, \quad t \in [0, 1].
\]

We will prove (8) by contradiction. Suppose there was no such constant. Then there would exist \( \{t_k\} \subset [0, 1], \{a_{n(k)}\}_{n,k \geq 0} \) and integers \( 0 < N_0 < N_1 < \ldots \) such that

\[
\left| \sum_{n=0}^{2^{N_k} - 1} a_{n(k)} w_n \right|_p = 1 \quad \text{and} \quad \left| \sum_{n=0}^{2^{N_k} - 1} a_{n(k)} e^{2\pi i n x} w_n \right|_p > 2^{2k}.
\]

Observe that for \( n = 0, 1, \ldots, 2^{N_k} - 1 \), \( n + 2^{N_k} = n + 2^{N_k} \in [2^{N_k}, 2^{N_k + 1}) \subset [2^{N_k}, 2^{N_k + 1}) \). Define a sequence \( \{\lambda_n\} \) by

\[
\lambda_n = \begin{cases} 
0 & \text{if } 0 < n < 2^{N_k}, \\
2^{-k} e^{2\pi i n x} & \text{if } 2^{N_k} < n < 2^{N_k + 1}, k = 0, 1, \ldots,
\end{cases}
\]

Then, for \( \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \in L^p \),

\[
\left| \sum_{n=0}^{\infty} \lambda_n a_n e^{2\pi i n x} \right|_p \leq \sum_{k=0}^{2^{N_k} - 1} 2^{-k} \left| \sum_{n=2^{N_k}}^{2^{N_k} + 1} a_n e^{2\pi i n x} \right|_p \\
\leq C_p \left| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \right|_p.
\]

(See [6, I, p. 266].) Hence \( \{\lambda_n\} \in M(L^p, \{e^{2\pi n i x}\}) \). On the other hand, for \( k = 0, 1, \ldots \),

\[
\left| \sum_{n=0}^{2^{N_k} - 1} a_{n(k)} w_{2^{N_k} + n} \right|_p = 1,
\]

whereas

\[
\left| \sum_{n=0}^{2^{N_k} - 1} a_{n(k)} \lambda_{2^{N_k} + n} w_{2^{N_k} + n} \right|_p = 2^{-k} \left| \sum_{n=0}^{2^{N_k} - 1} a_{n(k)} e^{2\pi i n x} w_n \right|_p > 2^k.
\]

By the closed graph theorem, \( \{\lambda_n\} \not\in M(L^p, \{w_n\}) \), contradicting our assumption. This proves (8). Similarly we can show that there is a constant \( C'_p \) such that for every \( \sum_{n=0}^{\infty} a_n e^{2\pi n i x} \in L^p \),

\[
(9) \quad \int \left| \sum_{n=0}^{\infty} a_n w_n(x) e^{2\pi i n x} \right|^p dx \leq C'_p \int \left| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \right|^p dx, \quad x \in [0, 1].
\]

We will now show that (8) and (9) imply the equivalence of \( \{w_n\} \) and \( \{e^{2\pi n i x}\} \). To see this, let \( a_1, \ldots, a_N \) be any numbers. From (8), we have

\[
\int \left| \sum_{n=0}^{N} a_n e^{2\pi i n x} w_n(x) \right|^p dx \leq C_p \int \left| \sum_{n=0}^{N} a_n w_n(x) \right|^p dx.
\]
From (9), we have
\[ \int \left| \sum_{n=0}^{N} a_n e^{2\pi i nx} \right|^p dt \leq C_p^p \int \left| \sum_{n=0}^{N} a_n \sin(x)e^{2\pi i nx} \right|^p dx. \]

Therefore \( \| \sum_{n=0}^{N} a_n e^{2\pi i nx} \|_p \leq C_p C_p \| \sum_{n=0}^{N} a_n \sin(x) \|_p. \) Similarly, we have
\[ \| \sum_{n=0}^{N} a_n \sin(x) \|_p \leq C_p C_p \| \sum_{n=0}^{N} a_n e^{2\pi i nx} \|_p. \]

This shows \( \{ w_n \} \) and \( \{ e^{2\pi inx} \}_{n \geq 0} \) are equivalent, which contradicts Theorem 2. This completes the proof of Theorem 3.

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Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903