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A NOTE ON WALSH-FOURIER SERIES

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ABSTRACT. It is shown that the double sequence $\{\lambda_{mn}\}$ with $\lambda_{mn} = 1$ if $n \leqslant m$ and 0 otherwise is an L^p multiplier for the Walsh system in two dimensions only if $p = 2$. This result is then used to show that the one-dimensional trigonometric system and the Walsh system are nonequivalent bases of the Banach space $L^p[0, 1]$, and hence have different L^p multipliers, $1 < p < \infty, p \neq 2$.

1. Let $\{\lambda_{mn}\}_{-\infty<m,n<\infty}$ be the double sequence defined by $\lambda_{mn} = 1$ if $n \leqslant m$ and 0 otherwise. For

$$f \sim \sum_{m,n=-\infty}^{\infty} a_{mn} e^{2\pi i(mx+ny)} \in L^p([0, 1] \times [0, 1]),$$

let

$$T_1 f \sim \sum_{m,n=-\infty}^{\infty} a_{mn}\lambda_{mn} e^{2\pi i(mx+ny)}.$$

It is well known that T_1 is bounded on $L^p([0, 1] \times [0, 1])$, $1 < p < \infty$. This is a consequence of the one-dimensional result of M. Riesz for the conjugate function [6, I, p. 253]. The L^p boundedness of T_1 was used, for example, in C. Fefferman's proof of the almost everywhere convergence of double Fourier series [1].

We now turn our attention to the Walsh system $\{w_n\}$. For

$$f \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y) \in L^p([0, 1] \times [0, 1]),$$

consider the corresponding operator T_2 defined by

$$T_2 f \sim \sum_{m,n=0}^{\infty} a_{mn}\lambda_{mn} w_m(x) w_n(y).$$

Because of the great similarity between the Walsh system and the trigonometric system, one would expect T_2 to be bounded on $L^p([0, 1] \times [0, 1])$, $1 < p < \infty$. However, this is not the case. In §2 we will show that T_2 is not bounded on L^p except for $p = 2$. This result is then used in §3 to give a negative answer to a question of P. Enflo: Are the trigonometric system and the Walsh system equivalent bases of the Banach space $L^p[0, 1]$, $1 < p < \infty$, $p \neq 2$? Finally in §4 we will deduce from the nonequivalence that the

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one-dimensional Walsh system and trigonometric system have different L^p multipliers, $1 < p < \infty, p \neq 2$.

2. Let $\{r_n\}_{n\geq 0}$ denote the Rademacher functions and $\{w_n\}_{n\geq 0}$ the Walsh functions defined on $I = [0, 1]$. For any two real numbers with dyadic expansions $a = \sum_{j=-\infty}^{\infty}\alpha_j 2^j$, $b = \sum_{j=-\infty}^{\infty}\beta_j 2^j$, α_j, $\beta_j = 0$ or 1, let $a \dot{+} b = \sum_{j=-\infty}^{\infty}|\alpha_j - \beta_j|2^j$. It is understood that we use the finite representation in the case of a dyadic rational. Basic properties of the Walsh functions can be found in [2].

THEOREM 1. *T_2 is bounded on $L^p(I^2)$ if and only if $p = 2$.*

PROOF. The case $p = 2$ is trivial by Parseval's formula. It is sufficient to show that T_2 is not bounded on L^p for $p < 2$, for then the theorem will follow by a duality argument.

Instead of dealing with T_2 we consider an equivalent operator T_2' defined on $L^p(I^2)$ as follows. Let $A = \{(m, n): 0 \leq n \leq m \dot{+} n, m = 0, 1, \ldots\}$. For $f \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y)$, let $T_2' f \sim \sum_{(m,n)\in A} a_{mn} w_m(x) w_n(y)$. Suppose g is the function on I^2 defined by $g(x, y) = f(x, x \dot{+} y)$. Since $w_n(x \dot{+} y) = w_n(x)w_n(y)$ and $w_{m\dot{+}n} = w_m w_n$, we have $T_2' g(x, y) = T_2 f(x, x \dot{+} y)$. Therefore the boundedness of T_2 is equivalent to that of T_2'.

We will define a sequence of functions $\{f_k\}$ on I^2 and show that

$$\|T_2' f_k\|_p / \|f_k\|_p \to \infty \quad \text{as } k \to \infty.$$

Let

$$f_k = \sum_{l=1}^{k} f_{kl}$$

where

$$f_{kl}(x, y) = 2^{-k} r_{l-1}(x) \prod_{j=0}^{k-1} (1 + r_j(y)).$$

We first note that

$$\|f_{kl}\|_p^p = \int \left| 2^{-k} \prod_{j=0}^{k-1} (1 + r_j(y)) \right|^p dy = 2^{-kp} \int_{[0,2^{-k}]} 2^{kp}\, dy = 2^{-k}. \tag{1}$$

Moreover, since $\int |\sum_{l=0}^{k-1} r_l(x)|^p\, dx \leq C_p^p k^{p/2}$ by Khintchin's inequality [6, I, p. 213], we have

$$\|f_k\|_p^p = \int \left| \sum_{l=0}^{k-1} r_l(x) \right|^p dx \int \left| 2^{-k} \prod_{j=0}^{k-1} (1 + r_j(y)) \right|^p dy \leq C_p^p k^{p/2} 2^{-k}, \tag{2}$$

where C_p denotes a constant depending only on p.

On the other hand, for $1 \leq l \leq k$,

$$T_2' f_{kl}(x, y) = 2^{-k} \sum_{n \leq m \dot{+} n; m} \int r_{l-1}(s) w_m(x)\, ds$$
$$\cdot \int \prod_{j=0}^{k-1} (1 + r_j(t)) w_n(t)\, dt\, w_m(x) w_n(y).$$

Now

$$\int r_{l-1}(s)w_m(s)\,dx = 1 \quad \text{if } m = 2^{l-1}$$

and 0 otherwise. Also

$$\int \prod_{j=0}^{k-1} (1 + r_j(t))w_n(t)\,dt = 1 \quad \text{if } n < 2^k$$
$$= 0 \quad \text{if } n \geqslant 2^k.$$

Therefore

$$T_2' f_{kl}(x,y) = 2^{-k} \sum_{n \leqslant 2^{l-1} \dot{+} n; n<2^k} w_n(y) w_{2^{l-1}}(x). \tag{3}$$

Let $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$ with $\varepsilon_j = 0$ or 1. We observe that $n \leqslant 2^{l-1} \dot{+} n$ if and only if $\varepsilon_{l-1} = 0$. Therefore

$$\begin{aligned} \sum_{n \leqslant 2^{l-1} \dot{+} n; n<2^k} w_n(y) &= \prod_{0 \leqslant j<k; j \neq l-1} (1 + r_j(y)) \\ &= \frac{1}{2}\left[\prod_{0 \leqslant j<k} (1 + r_j(y)) + \prod_{0 \leqslant j<k} \left(1 + r_j(y \dot{+} 2^{-l})\right)\right]. \end{aligned} \tag{4}$$

From (3) and (4), we have

$$T_2' f_{kl}(x,y) = \tfrac{1}{2}\left[f_{kl}(x,y) + f_{kl}(x, y \dot{+} 2^{-l})\right],$$

and hence

$$T_2' f_k(x,y) = \frac{1}{2} f_k(x,y) + \frac{1}{2} \sum_{l=1}^{k} f_{kl}(x, y \dot{+} 2^{-l}).$$

Since $f_k(x, y)$ and $f_{kl}(x, y \dot{+} 2^{-l})$, $l = 1, \ldots, k$, have mutually disjoint supports, it follows from (1) that

$$\|T_2' f_k\|_p^p \geqslant 2^{-p} \sum_{l=1}^{k} \|f_{kl}\|_p^p = 2^{-p} k 2^{-k}. \tag{5}$$

Combining (2) and (5), we obtain, for $p < 2$,

$$\|T_2' f_k\|_p / \|f_k\|_p \geqslant 2^{-1} C_p^{-1} k^{(1/p - 1/2)} \to \infty \quad \text{as } k \to \infty.$$

This completes the proof of Theorem 1.

3. It is known that $\{\cos \pi n x\}$ and $\{w_n\}$ are bases of the Banach space $L^p(I)$, $1 < p < \infty$. (See [6, I, p. 266] and [4].) We say that the sequences $\{u_n\}$, $\{v_n\}$ of a Banach space are equivalent if for every sequence of numbers $\{a_n\}$, $\sum_{n=0}^{\infty} a_n u_n$ converges if and only if $\sum_{n=0}^{\infty} a_n v_n$ converges. R. Askey, S. Wainger and J. E. Gilbert showed that $\{\cos \pi n x\}$ and certain classical orthonormal sequences are equivalent in $L^p(I)$, $1 < p < \infty$. (See [3].) We have the following

THEOREM 2. *Let* $1 < p < \infty$. $\{\cos \pi n x\}$ *and* $\{w_n\}$ *are equivalent bases of* $L^p(I)$ *if and only if* $p = 2$.

PROOF. Again the case $p = 2$ is trivial by Parseval's formula. Suppose they

were equivalent in $L^p(I)$, $p \neq 2$. From this it would follow that $\{e^{2\pi inx}\}_{n\geqslant 0}$ and $\{w_n\}$ are also equivalent in $L^p(I)$. (See [6, I, p. 253].) By the Banach-Steinhaus theorem, there exist constants C_p, $C_p' > 0$ such that for any sequence of numbers $\{a_n\}$,

$$(6)\qquad C_p^{-1}\left\|\sum_{n=0}^{N} a_n e^{2\pi inx}\right\|_p \leqslant \left\|\sum_{n=0}^{N} a_n w_n\right\|_p \leqslant C_p'\left\|\sum_{n=0}^{N} a_n e^{2\pi inx}\right\|_p, \qquad N \geqslant 0.$$

(See [5, p. 70].) Let $\{a_{mn}\}$ be any double sequence of numbers. Applying (6) first to the x-variable and then to the y-variable, we obtain

$$(7)\qquad \begin{aligned} C_p^{-2}\left\|\sum_{m,n=0}^{N} a_{mn} e^{2\pi i(mx+ny)}\right\|_p &\leqslant \left\|\sum_{m,n=0}^{N} a_{mn} w_m(x) w_n(y)\right\|_p \\ &\leqslant C_p'^2\left\|\sum_{m,n=0}^{N} a_{mn} e^{2\pi i(mx+ny)}\right\|_p, \qquad N \geqslant 0. \end{aligned}$$

Now, it follows from the corresponding one-dimensional result that for any function in $L^p(I^2)$, the square partial sums of both its trigonometric Fourier series and Walsh-Fourier series converge in $L^p(I^2)$. Therefore (7) implies the following: for any sequence of numbers $\{a_{mn}\}$, $f \sim \sum_{m,n=0}^{\infty} a_{mn} e^{2\pi i(mx+ny)}$ for some $f \in L^p(I^2)$ if and only if $g \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y)$ for some $g \in L^p(I^2)$.

Suppose $g \in L^p(I^2)$ with $g \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y)$. Then $f \sim \sum_{m,n=0}^{\infty} a_{mn} e^{2\pi i(mx+ny)} \in L^p(I^2)$. Hence

$$T_1 f \sim \sum_{m,n=0}^{\infty} a_{mn}\lambda_{mn} e^{2\pi i(mx+ny)} \in L^p(I^2),$$

which implies

$$T_2 g \sim \sum_{m,n=0}^{\infty} a_{mn}\lambda_{mn} w_m(x) w_n(y) \in L^p(I^2).$$

Therefore T_2 maps $L^p(I^2)$ into $L^p(I^2)$. By the closed graph theorem, T_2 is bounded on $L^p(I^2)$, contradicting Theorem 1. This proves Theorem 2.

4. Let $\{u_n\}$ be one of the sequences $\{w_n\}$, $\{\cos \pi nx\}$ or $\{e^{2\pi inx}\}_{n\geqslant 0}$. $M(L^p, \{u_n\})$ denotes the collection of all sequences $\{\lambda_n\}$ such that $f \sim \sum_{n=0}^{\infty} a_n u_n \in L^p(I)$ implies $g \sim \sum_{n=0}^{\infty} \lambda_n a_n u_n \in L^p(I)$. We will deduce from Theorem 2 the following

THEOREM 3. $M(L^p, \{\cos \pi nx\}) \neq M(L^p, \{w_n\})$, $1 < p < \infty$, $p \neq 2$.

We note that in general two nonequivalent bases of $L^p(I)$ may have the same multipliers. See, for example, [5, p. 484 and p. 546].

PROOF. Suppose they were equal. Since

$$M(L^p, \{\cos \pi nx\}) = M\left(L^p, \{e^{2\pi inx}\}_{n>0}\right),$$

we have $M(L^p, \{w_n\}) = M(L^p, \{e^{2\pi inx}\}_{n\geqslant 0})$. Let $\sum_{n=0}^{\infty} a_n e^{2\pi inx} \in L^p$. For every $t \in [0, 1]$,

$$\left\|\sum_{n=0}^{\infty} a_n e^{2\pi int} e^{2\pi inx}\right\|_p = \left\|\sum_{n=0}^{\infty} a_n e^{2\pi inx}\right\|_p,$$

so $\{e^{2\pi int}\} \in M(L^p, \{e^{2\pi inx}\})$, and hence belongs to $M(L^p, \{w_n\})$. We assert that, moreover, there is a constant C_p, depending only on p, such that for every $\sum_{n=0}^{\infty} a_n w_n \in L^p$,

$$(8) \qquad \int \left| \sum_{n=0}^{\infty} a_n e^{2\pi int} w_n(x) \right|^p dx \leqslant C_p^p \int \left| \sum_{n=0}^{\infty} a_n w_n(x) \right|^p dx, \qquad t \in [0, 1].$$

We will prove (8) by contradiction. Suppose there was no such constant. Then there would exist $\{t_k\} \subset [0, 1]$, $\{a_n^{(k)}\}_{n,k \geqslant 0}$ and integers $0 \leqslant N_0 < N_1 < \ldots$ such that

$$\left\| \sum_{n=0}^{2^{N_k}-1} a_n^{(k)} w_n \right\|_p = 1 \quad \text{and} \quad \left\| \sum_{n=0}^{2^{N_k}-1} a_n^{(k)} e^{2\pi int_k} w_n \right\|_p \geqslant 2^{2k}.$$

Observe that for $n = 0, 1, \ldots, 2^{N_k} - 1$, $n \dotplus 2^{N_k} = n + 2^{N_k} \in [2^{N_k}, 2^{N_k+1}) \subset [2^{N_k}, 2^{N_{k+1}})$. Define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \begin{cases} 0 & \text{if } 0 \leqslant n < 2^{N_0}, \\ 2^{-k} e^{2\pi int_k} & \text{if } 2^{N_k} \leqslant n < 2^{N_{k+1}}, k = 0, 1, \ldots. \end{cases}$$

Then, for $\sum_{n=0}^{\infty} a_n e^{2\pi inx} \in L^p$,

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} \lambda_n a_n e^{2\pi inx} \right\|_p &\leqslant \sum_{k=0}^{\infty} 2^{-k} \left\| \sum_{n=2^{N_k}}^{2^{N_{k+1}}-1} a_n e^{2\pi inx} \right\|_p \\ &\leqslant C_p \left\| \sum_{n=0}^{\infty} a_n e^{2\pi inx} \right\|_p. \end{aligned}$$

(See [6, I, p. 266].) Hence $\{\lambda_n\} \in M(L^p, \{e^{2\pi inx}\})$. On the other hand, for $k = 0, 1, \ldots,$

$$\left\| \sum_{n=0}^{2^{N_k}-1} a_n^{(k)} w_{2^{N_k}+n} \right\|_p = 1,$$

whereas

$$\left\| \sum_{n=0}^{2^{N_k}-1} a_n^{(k)} \lambda_{2^{N_k}+n} w_{2^{N_k}+n} \right\|_p = 2^{-k} \left\| \sum_{n=0}^{2^{N_k}-1} a_n^{(k)} e^{2\pi int_k} w_n \right\|_p \geqslant 2^k.$$

By the closed graph theorem, $\{\lambda_n\} \not\in M(L^p, \{w_n\})$, contradicting our assumption. This proves (8). Similarly we can show that there is a constant C_p' such that for every $\sum_{n=0}^{\infty} a_n e^{2\pi int} \in L^p$,

$$(9) \qquad \int \left| \sum_{n=0}^{\infty} a_n w_n(x) e^{2\pi int} \right|^p dt \leqslant C_p'^p \int \left| \sum_{n=0}^{\infty} a_n e^{2\pi int} \right|^p dt, \qquad x \in [0, 1].$$

We will now show that (8) and (9) imply the equivalence of $\{w_n\}$ and $\{e^{2\pi inx}\}$. To see this, let $a_1, \ldots, a_N$ be any numbers. From (8), we have

$$\iint \left| \sum_{n=0}^{N} a_n e^{2\pi int} w_n(x) \right|^p dx dt \leqslant C_p^p \int \left| \sum_{n=0}^{N} a_n w_n(x) \right|^p dx.$$

From (9), we have

$$\int \left| \sum_{n=0}^{N} a_n e^{2\pi int} \right|^p dt \leqslant C_p'^p \iint \left| \sum_{n=0}^{N} a_n w_n(x) e^{2\pi int} \right|^p dt dx.$$

Therefore $\|\sum_{n=0}^{N} a_n e^{2\pi int}\|_p \leqslant C_p C_p' \|\sum_{n=0}^{N} a_n w_n\|_p$. Similarly, we have

$$\left\| \sum_{n=0}^{N} a_n w_n \right\|_p \leqslant C_p C_p' \left\| \sum_{n=0}^{N} a_n e^{2\pi int} \right\|_p.$$

This shows $\{w_n\}$ and $\{e^{2\pi inx}\}_{n \geqslant 0}$ are equivalent, which contradicts Theorem 2. This completes the proof of Theorem 3.

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