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ON THE HULL OF A LINEAR DIFFERENTIAL OPERATOR

MAYER HUMI

ABSTRACT. For a system of linear differential equations with almost periodic coefficients with respect to a group G we generalize previous results by Bochner and Sell. Furthermore we explore the relation between the solutions of the original system and those in its hull.

I. **Introduction.** In 1962 Bochner [1] considered the existence of almost periodic (a.p.) solutions to linear ordinary differential equations with a.p. coefficients. More recently Sell [2] discussed the same problem for linear partial differential equations. However, if we examine these two papers from a group theoretical point of view it is easy to realize that T_m (= the group of translations in R^m) plays a distinguished role in them. Thus almost periodicity was defined only with respect to (wrt) translations and the operators

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}}, \qquad i_1 + \cdots + i_m = \alpha,$$

are invariant wrt T_m.

In §II of this paper we generalize the definitions and results of [1], [2] to any group G of affine transformations on R^n [3]. In §III we investigate the relationship between the solutions of a given a.p. differential operator and those in its hull (see definition §II) while in §IV we investigate the subgroup structure of the hull. Finally in §V we give some examples.

II. **Almost periodicity in R^n.** In this section we generalize the results of [1], [2].

DEFINITION 1. A function $f: R^m \to R^n$ is said to be C^k-bounded if $f = (f_1 \cdots f_n)$ together with all its derivatives up to and including order k are bounded and uniformly continuous on R^m.

DEFINITION 2. Let G be a group of affine transformations acting on R^m. We say that $f: R^m \to R$ is a.p. wrt G if for any sequence $\{g'_k\}$, $g'_k \in G$, there exists a subsequence $\{g_k\}$ such that $\lim_{k\to\infty} f(g_k t)$ exists and is uniform in R^m.

DEFINITION 3. A system of linear differential equations on R^m (ordinary or partial) is a set

$$\sum L_{ij} u_j = f_i \tag{2.1}$$

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where L_{ij} is an arbitrary linear differential operator on R^m,

$$L_{ij} = \sum a_{\alpha ij} D^{\alpha} \tag{2.2}$$

(the summation is finite).

In the following we denote such a system by (L, f) or $Lu = f$.

DEFINITION 4. A system of linear differential equations $Lu = f$ on R^m is said to be a.p. with respect to G if D^{α} *are invariant differential operators* wrt G [3] and $a_{\alpha ij}$, f_i are a.p. wrt G.

From this definition it is clear that the operators D^{α} will depend on the group G under consideration. If $G = T_m$ then D^{α} are those defined in §I, but they will be different if we consider other groups.

DEFINITION 5. Let (L, f) be a.p. wrt G. The hull $H(L, f, G)$ of (L, f) wrt G is the collection of all systems $L^*u = f^*$ where the functions $a^*_{\alpha ij}$, f^*_i are related to $a_{\alpha ij}$, f_i by

$$\begin{aligned} \lim_{k\to\infty} a_{\alpha ij}(g_k t) &= a^*_{\alpha ij}(t), \\ \lim_{k\to\infty} f_i(g_k t) &= f^*_i(t), \end{aligned} \qquad g_k \in G,\ t \in R^m, \tag{2.3}$$

for some sequence $g = \{g_k\}$ which is independent of α, i, j (this limit is pointwise).

REMARKS. 1. In the following we denote the operation that transforms (L, f) into (L^*, f^*) by $T(g)$ and write $T(g)L = L^*$, $T(g)f = f^*$.

2. In the following almost periodicity will always mean almost periodicity wrt a group G acting on R^m.

To generalize the results in [1], [2] we start by observing that if f is a.p. then so is $f(gt)$, $g \in G$. Moreover if $\{f_n(t)\}$ is a sequence of a.p. functions which converges uniformly then the limit function is again a.p. From these remarks we obtain the following lemma which is a generalization of Theorem 1 in [1].

LEMMA 1. *If f is a.p. wrt G and $\{g'_k\}$, $\{h'_k\}$ are any infinite sequences in G then there are infinite subsequences,*

$$\{g_l\} = \{g'_{k_l}\}, \qquad \{h_l\} = \{h'_{k_l}\},$$

for a common sequence of indices $\{k_l\}$ such that

$$\lim_{s\to\infty} \lim_{r\to\infty} f(g_r h_s t) = \lim_{k\to\infty} f(g_k h_k t). \tag{2.4}$$

COROLLARY 1. *If $f(t)$ is a.p. then any infinite sequence $\{g'_k\}$ contains an infinite subsequence $\{g_k\}$ for which*

$$\lim_{s\to\infty} \lim_{r\to\infty} f(g_r g_s^{-1} t) = f(t). \tag{2.5}$$

Using the operator $T(g)$ we can rewrite these results in the form

$$T(g)T(h)f = T(gh)f, \tag{2.6}$$

$$T(g)T(g^{-1})f = f. \tag{2.7}$$

DEFINITION 6. We say that G acts effectively on R^n if for any $x, y \in R^n$

there exists $g \in G$ such that $gx = y$ (i.e. R^n is a homogeneous space of G).

From now on all groups G considered will act effectively on R^n. We now prove the inverse of Lemma 1 (the counterpart of Theorem 1 in [1]).

THEOREM 1. *If f is such that any two infinite sequences in G contain infinite subsequences for which the equality*

$$\lim_{s\to\infty} \lim_{r\to\infty} f(\alpha_r \beta_s t) = \lim_{k\to\infty} f(\alpha_k \beta_k t) \tag{2.8}$$

holds and where each of the three limits is assumed to exist for every t, then $f(t)$ is a.p.

PROOF. If we put $\alpha_k' = \gamma_k'$, $\beta_k' = e$ (the unit element in G) then our assumptions imply that every infinite sequence $\{\gamma_k'\}$ contains an infinite subsequence $\{\gamma_k\}$ such that $\lim_{k\to\infty} f(\gamma_k t) = g(t)$ exists for every t. We now show that the limit is uniform. If this were not true then nonuniformity would imply that for some $\varepsilon_0 > 0$ there exist monotone sequences of integers $\{p_k\}$, $\{q_k\}$ and a sequence of points $\{\delta_l\}$, $\delta_l \in R^n$, such that

$$|f(\gamma_{p_k}\delta_k) - f(\gamma_{q_k}\delta_k)| \geqslant \varepsilon_0. \tag{2.9}$$

However since G is effective on R^n there exists $\{\beta_l\}$ such that $\delta_l = \beta_l \cdot \mathbf{0}$. The assumptions of the theorem imply that the following limits exist (through the choice of subsequences if necessary)

$$\lim_{s\to\infty} \lim_{r\to\infty} f(\gamma_{p_r}\beta_s \cdot \mathbf{0}) = f_p(\mathbf{0}), \tag{2.10}$$

$$\lim_{s\to\infty} \lim_{r\to\infty} f(\gamma_{q_r}\beta_s \cdot \mathbf{0}) = f_q(\mathbf{0}), \tag{2.11}$$

but $\{p_r\}$, $\{q_r\}$ are subsequences of the original sequence, and therefore,

$$g(\delta_s) = \lim_{r\to\infty} f(\gamma_{p_r}\beta_s \cdot \mathbf{0}) = \lim_{r\to\infty} f(\gamma_{q_r}\beta_s \cdot \mathbf{0}).$$

this implies that

$$f_p(\mathbf{0}) = f_q(\mathbf{0}) = \lim_{s\to\infty} g(\delta_s)$$

while (2.9) implies

$$|f_p(\mathbf{0}) - f_q(\mathbf{0})| \geqslant \varepsilon_0 > 0$$

which is a contradiction.

DEFINITION 7. A continuous function f: $R^m \to R^n$ is said to be almost automorphic (a.a.) if for every sequence β' there is a subsequence β such that the limit $T(\beta)f$ exists and $T(\beta^{-1})T(\beta)f = f$.

Using this definition we can define an a.a. system of differential equations and the hull of such a system in exactly the same way that these concepts were introduced for the a.p. case.

With these preparations we can generalize the following results of [1], [2] to any group G which acts effectively on R^m. The proofs of these results are exactly the same as in [2] except for the generalized meaning of the operators

$T(g)$ (and the replacement of the $+$ sign by the group multiplication).

LEMMA 2 (SELL). *Let f: $R^m \to R^n$ be a continuous function. Then f is a.p. wrt G if and only if for any two sequences β', γ' there exist subsequences β, γ such that*

(1) *$T(\beta)T(\gamma)f$ is a.a. wrt G,*
(2) *$Bf = f$,*
where $B = T((\beta\gamma)^{-1})T(\beta)T(\gamma)$.

LEMMA 3 (BOCHNER-SELL). *Let L be a linear differential operator of order k in R^n which is a.a. wrt G. If every C^k-bounded solution of the equation $L^*v = 0$ for any $L^* \in H(L, 0, G)$ is a.a. wrt G then every C^k-bounded solution of $L^*u = f^*$. $(L^*, f^*) \in H(L, f, G)$ is a.a. wrt G.*

THEOREM 2. *Let L be a linear differential operator of order k which is a.p. wrt G. If every C^k-bounded solution of the equation*

$$L^*v = 0 \quad \text{for any } L^* \in H(L, 0, G)$$

is a.p. wrt G then every C^k-bounded solution of

$$L^*u = f^*, \qquad (L^*, f^*) \in H(L, f, G),$$

is a.p. wrt G.

III. **The structure of the hull**. In this section the phrases a.p. and a.a. will always mean a.p. and a.a. wrt G where G acts effectively on R^n. Similarly (L, f) will denote a system of linear differential equations which is a.p. wrt G.

LEMMA 4. *If (L, f) has a C^k-bounded a.p. solution then every $(L^*, f^*) \in H(L, f, G)$ has a C^k-bounded a.p. solution.*

PROOF. By definition there exists $T(\gamma)$ such that

$$T(\gamma)L = L^*, \qquad T(\gamma)f = f^*. \tag{3.1}$$

Let u be an a.p. C^k-bounded solution of (L, f); then

$$T(\gamma)(Lu) = T(\gamma)f, \tag{3.2}$$

and by taking a subsequence if necessary [1] we obtain that

$$L^*(T(\gamma)u) = f^*. \tag{3.3}$$

Thus $T(\gamma)u$ is a solution of (L^*, f^*). Moreover $T(\gamma)u$ is an a.p. C^k-bounded solution as a limit of a C^k-bounded a.p. function.

LEMMA 5. *If all C^k-bounded solutions of (L, f) are a.p. then any a.a. solution u^* of (L^*, f^*) is a.p. where $(L^*, f^*) \in H(L, f, G)$.*

PROOF. By definition there exists $T(\gamma)$ such that (3.1) is satisfied. However, (L^*, f^*) is a.p. and therefore $T(\gamma^{-1})L^* = L$, $T(\gamma^{-1})f^* = f$. Therefore

$$T(\gamma^{-1})(L^*u^*) = L\big(T(\gamma^{-1})u^*\big)$$

from which we infer that $T(\gamma^{-1})u^*$ is a solution of (L, f) and hence a.p. But u^* is a.a. and therefore

$$u^* = T(\gamma)T(\gamma^{-1})u^* = T(\gamma)\big(T(\gamma^{-1})u^*\big)$$

which proves our statement.

DEFINITION 8. By $P(L, f)$ we denote the space of all C^k-bounded a.p. solutions of (L, f).

THEOREM 3. *If* $\dim P(L, f) = r$ *(finite or infinite) then* $\dim P(L^*, f^*) = r$ *for all* $(L^*, f^*) \in H(L, f, G)$.

PROOF. If $r < \infty$ let $\{u_i\}_1^r$ be a basis for $P(L, f)$ and let

$$(L^*, f^*) \in H(L, f, G);$$

then there exist $T(\gamma)$ such that $T(\gamma)L = L^*$, $T(\gamma)f = f^*$, and therefore by taking subsequences if necessary,

$$f^* = T(\gamma)(Lu_i) = L^*(T(\gamma)u_i)$$

for all $1 \leqslant i \leqslant r$. Thus $T(\gamma)u_i \in P(L^*, f^*)$. We claim that $\{T(\gamma)u_i\}_1^r$ are independent. In fact if $\sum c_i T(\gamma)u_i = 0$ then

$$0 = T(\gamma^{-1})\Big(\sum c_i T(\gamma)u_i\Big) = \sum c_i T(\gamma^{-1})T(\gamma)u_i = \sum c_i u_i$$

from which we infer that all $c_i = 0$. Thus $\dim P(L^*, f^*) \geqslant \dim P(L, f)$. But $L \in H(L^*, f^*, G)$ and therefore $\dim P(L, f) \geqslant \dim P(L^*, f^*)$ and thus $\dim P(L, f) = \dim P(L^*, f^*)$.

If $r = \infty$ then the proof above implies that for any r, $\dim P(L^*, f^*) \geqslant r$ and thus $\dim P(L^*, f^*) = \infty$.

COROLLARY. *If (L, f) is a linear ordinary differential operator of order k whose solutions are C^k-bounded and a.p., then all solutions of*

$$(L^*, f^*) \in H(L, f, G)$$

are C^k-bounded and a.p.

IV. **Subgroup structure of the hull.** Let A be a group of linear transformations acting effectively on R^n and (L, f) a system of linear differential equations which is a.p. wrt A. In the following we denote by G, K subgroups of A such that $GK = KG$.

LEMMA 6. *If u is a.p. wrt both G and K then for any sequence $\{g_n\}$, $g_n \in G$, such that $v(t) = \lim_{n\to\infty} u(g_n t)$ exists, the limit function is a.p. wrt K.*

PROOF. Let $\{k'_n\}$, $k'_n \in K$, be given. We must show that there exists a subsequence $\{k_n\}$ such that

$$\lim_{s\to\infty} v(k_s t) = \lim_{s\to\infty} \lim_{r\to\infty} u(k_s g_r t)$$

exists and is uniform in t. However this is a direct consequence of the fact that the separate limits are uniform in t by the assumption.

COROLLARY. *If all C^k-bounded solutions of (L, f) are a.p. wrt K and G then they are a.p. wrt $B = GK$.*

PROOF. Let $\{b'_n\}$, $b'_n \in B$, be given. Since $B = GK$, $b'_n = g'_n k'_n$. By the previous lemma we infer that there exist subsequences of $\{k'_n\}$, $\{g'_n\}$ with a common set of indices such that $\lim_{s\to\infty} \lim_{r\to\infty} u(g_s k_r t)$ exists and is uniform in t. It then follows that

$$\lim_{s\to\infty} \lim_{r\to\infty} u(g_s k_r t) = \lim_{n\to\infty} u(g_n k_n t) = \lim_{n\to\infty} u(b_n t)$$

exists and is uniform in t, which proves our statement.

LEMMA 7. *Let u^* be a C^k-bounded solution of $(L^*, f^*) \in H(L, f, K)$ which is a.p. wrt K. If all C^k-bounded solutions of (L, f) are a.p. wrt G and K, then u^* is a.p. wrt G.*

PROOF. Since $(L^*, f^*) \in H(L, f, K)$ there exists $T(k^{-1})$, $k = \{k_n\}$, $k_n \in K$, such that $T(k^{-1})L^* = L$, $T(k^{-1})f^* = f$. This implies that

$$f = T(k^{-1})(L^* u^*) = L\big(T(k^{-1})u^*\big);$$

thus $v = T(k^{-1})u^*$ is a solution of (L, f) and hence a.p. wrt GK. Therefore

$$T(k)v = T(k)T(k^{-1})u^* = u^*$$

is also a.p. wrt G by Lemma 6.

In the following, by a solution of (L, f) we mean a C^k-bounded solution.

THEOREM 4. *Let $(L^*, f^*) \in H(L, f, G)$ and $(\bar{L}, \bar{f}) \in H(L^*, f^*, K)$. If all solutions of any (L, f) are a.p. wrt K, then all solutions of any*

$$(\tilde{L}, \tilde{f}) \in H(L, f, GK)$$

are a.p. wrt GK.

PROOF. Let $(\tilde{L}, \tilde{f}) \in H(L, f, GK)$; then there exists $T(gk)$ such that

$$T(gk)L = T(g)T(k)L = \tilde{L}, \qquad T(gk)f = T(g)T(k)f = \tilde{f}.$$

If we denote $T(g)L$, $T(g)f$ by L^*, f^*, respectively, then

$$(L^*, f^*) \in H(L, f, G).$$

By the assumption of the theorem and Lemma 7 we infer that all its C^k-bounded solutions are a.p. wrt G and K. However, $(\tilde{L}, \tilde{f}) \in H(L^*, f^*, K)$ and, by the assumptions, all C^k-bounded solutions of $(\tilde{L}, \tilde{f})$ are a.p. wrt K. Lemma 7 now implies that all C^k-bounded solutions of $(\tilde{L}, \tilde{f})$ are a.p. wrt G and K. By the corollary following Lemma 6 this is equivalent to being a.p. wrt GK.

As an application of this theorem we consider the following

COROLLARY. *Let (L, f) be a.p. wrt T_m. If all the solutions of any $(L^*, 0) \in H(L, 0, T_m)$ are a.p. wrt T_m and all the solutions of any*

$$(\bar{L}, 0) \in H(L^*, 0, T_l)$$

are a.p. wrt T_l *(where* $k + l = m$*), then all solutions of any operator in* $H(L, f, T_m)$ *are a.p. wrt* T_m.

PROOF. Direct application of Theorem 4 above and Theorem 1 in [2].

V. **Examples.** In this section we give some examples of operators to which the theory given above can be applied.

1. Let $G = O(3)$ be the group of rotations in three dimensions; then G is a group of linear transformations acting on R^3 (*not effectively*). The Laplace operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is an invariant operator wrt G. Therefore an operator in which $a_{\alpha ij}$, f_i are a.p. wrt G and $D^\alpha = \nabla^2 \cdots \nabla^2$ (α-times) is an a.p. operator wrt G.

2. Let $G = E(2)$ be the Euclidean group in two dimensions which consists of rotations and translations in the plane R^2. G acts effectively on R^2, and the operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is an invariant operator of G.

REMARK. These two examples can be extended naturally to any R^n. However for higher dimensions the corresponding groups $O(n)$, $E(n)$ have more than one independent invariant operator (these are the Casimir operators of the group) so that D^α can then be any monomial in these operators [4].

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DEPARTMENT OF MATHEMATICS, WORCESTER POLYTECHNIC INSTITUTE, WORCESTER, MASSACHUSETTS 01609