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ON A UNIQUENESS THEOREM FOR ANALYTIC MAPPINGS

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ABSTRACT. A generalization of the uniqueness theorem of Koebe-Nevanlinna for meromorphic functions of bounded type is given on Riemann surfaces.

Koebe-Nevanlinna's theorem (cf. [2, pp. 41–42]) is stated as follows:

Let $\phi(z)$ be a meromorphic function of bounded type in $\{|z| < 1\}$. Let there be a sequence $\{\alpha_n\}$ of arcs each contained in an annulus $\{1 - \varepsilon_n < |z| < 1\}$, $\lim_{n\to\infty}\varepsilon_n = 0$, having as limiting set an arc of $\{|z| = 1\}$ and such that on α_n, $|\phi(z) - c| < \eta_n$, $\lim_{n\to\infty}\eta_n = 0$. Then $\phi(z) \equiv c$.

Let f be an analytic mapping of a hyperbolic Riemann surface R into a closed or a parabolic Riemann surface R'. Let R^* and R'^* denote the Martin compactification and the Kerékjártó-Stoïlow compactification of R and R', respectively. $\overline{X}$ means the closure of a set X ($\subset R^*$ or R'^*) with respect to R^* or R'^*.

Let Δ_1 denote the set of minimal points in $\Delta = R^* - R$. For $p \in \Delta_1$, let F_p be a filter basis on R with respect to the fine topology, and set $f^{\frown}(p) = \bigcap_{G\in F_p}\overline{f(G)}$. When $f^{\frown}(p)$ consists of a single point, $f^{\frown}(p)$ is denoted by $\hat{f}(p)$. We set $F(f) = \{q \in \Delta_1; f^{\frown}(q) = \hat{f}(q)\}$ and for $p \in \Delta$,

$$U_n(p) = \{a \in R; d(p, a) < 1/n\},$$

where d denotes a metric on R^*. For $b \in \Delta' = R'^* - R'$, let $\{G_n(b)\}$ be a determinant sequence of b. For $b \in \Delta'$, we set $V_n(b) = G_n(b)$ and for $b \in R'$, $V_n(b) = \{b' \in R'; d'(b, b') < 1/n\}$, where d' denotes a metric on R'.

Let $\{\lambda_n\}$ be a sequence of Jordan arcs such that each λ_n is compact on R. It is said that $\{\lambda_n\}$ converges to the ideal boundary of R, when for every compact set $K \subset R$ there exists an $n(K)$ such that $\lambda_n \subset R - K$ for all $n \geqslant n(K)$. We say that $p \in \Delta_1$ is an accumulating point of a set $X \subset R$ with respect to the fine topology, when for any $G \in F_p$, $G \cap X \neq \varnothing$. Further it is called that $\{f(\lambda_n)\}$ converges to a point $c \in R'^*$, when there exists an $n(m)$ such that for every m, $f(a) \subset V_m(c)$ on λ_n for all $n \geqslant n(m)$.

LEMMA 1. *Δ' is a polar set.*

PROOF. For a closed parametric disk K_0 in R', we set $R'_0 = R' - K_0$. Let

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$g_c(b)$ be the Green function of R'_0 with pole at a point $c \in R'_0$.

Since R'_0 is a parabolic end, from the maximum principle G.H.M. $g_c(b) \equiv 0$ and $\lim \inf_n g_c(b_n) > 0$ for any sequence $\{b_n\}$ of points converging to a point $e \in \Delta'$. Here "G.H.M." means "greatest harmonic minorant".

Therefore there exists a positive superharmonic function s on R'_0 with $\lim_n s(b_n) = \infty$ (cf. [1, p. 27]). Hence $\lim_{b \to e} s(b) = \infty$ for each point $e \in \Delta'$.

LEMMA 2 (CF. [1, pp. 146–147]). *For $p \in F(f)$, let $D_n \in F_p$ be the connected component of $U_n(p) \cap f^{-1}(V_n(\hat{f}(p)))$. Then there exists a Jordan arc γ: $a = \gamma(t)$ $(0 \leqslant t < 1)$ converging to p as follows: $\lim_{a \to p; a \in \gamma} f(a) = \hat{f}(p)$ and for every n there exists a $t(n)$ such that $\gamma(t) \subset D_n$ for all $t \geqslant t(n)$.*

PROOF. We can see easily that $D_{n+1} \subset D_n$. Take a point $a_1 \in D_1$, and then choose a $D_{k_2} \in \{D_n\}$ satisfying $a_1 \not\in D_{k_2}$. There exists a piecewise analytic Jordan arc γ_1 joining a_1 and a point $a_2 \in D_{k_2}$ in D_1 such that $\gamma_1 \cap \partial D_{k_2}$ consists of a single point, where ∂D_{k_2} denotes the relative boundary of D_{k_2} on R.

Next choose a $D_{k_3} \in \{D_n\}$ satisfying $D_{k_3} \cap \gamma_1 = \varnothing$. Since $D_{k_2} - \gamma_1$ is a region, there exists a piecewise analytic Jordan arc γ_2 joining a_2 and a point $a_3 \in D_{k_3}$ in $D_{k_2} - \gamma_1$ such that $\gamma_2 \cap \partial D_{k_3}$ consists of a single point. By repeating the same method, we have a desired Jordan arc γ.

For $p \in F(f)$, let Γ_p denote the family of the Jordan arc γ in Lemma 2. Γ_p is determined independently of the choice of countable fundamental neighborhood systems at p and $\hat{f}(p)$, respectively.

THEOREM. *Let f be a Fatou mapping and let a set $A \subset \Delta$ be of harmonic measure positive. Let $\{\lambda_n\}$ converge to the ideal boundary of R and let $\{f(\lambda_n)\}$ converge to a point $c \in R'^*$. If for each $p \in A \cap \Delta_1$ there exists a $\gamma \in \Gamma_p$ such that $\gamma \cap \lambda_n \neq \varnothing$ for each n, then f is constant.*

PROOF. Let $\{b'_i\}$ be a countable dense set in R' and $\{b''_i\}$ all the points of Δ', then we consider the sequence $\{b_i\}$ such that $b'_1, b''_1, b'_2, b''_2, \ldots, b'_k, b''_k, \ldots$. For $V_{m'}(b_m)$ and $V_{n'}(b_n)$ $(b_m \neq b_n)$ satisfying

$$\overline{V_{m'}(b_m)} \cap \overline{V_{n'}(b_n)} = \varnothing,$$

a countable family $\{h\}$ of bounded continuous functions on R'^* with $h = m''$ on $\overline{V_{m'}(b_m)}$ and with $h = n''$ on $\overline{V_{n'}(b_n)}$ can be constructed.

Therefore there exist h' and h'' $(\in \{h\})$ such that $h'(b')h''(b'') - h'(b'')h''(b') \neq 0$ for any points b' and b'' $(b' \neq b'')$ in R'^*. Since R'^* is a resolutive compactification, as we see from [1, pp. 152–153] we get [1, Theorem 14.4]. Hence there exists a set A' of harmonic measure positive such that $A' \subset F(f) \cap A$.

For any $q \in A'$ and any $U_n(q)$, there exists a point a^* satisfying $a^* \in U_n(q) \cap \gamma \cap \lambda_M$ for some M. Indeed if not, there exists a $U_N(q)$ such that $\gamma \cap \lambda_n \subset R - U_N(q)$ for every n. Hence for a compact set $K = \gamma \cap (R - U_N(q))$ and every n, we have $K \cap \lambda_n \neq \varnothing$. This is a contradiction.

Therefore we can take a sequence $\{a_n\}$ of $a_n \in \gamma$ such that $\{a_n\}$ converges to q and $\{f(a_n)\}$ converges to c.

Suppose that f is nonconstant. Since $\hat{f}(q) = \lim_{a \to q; a \in \gamma} f(a) = c$, from Lemma 1 and [1, Theorem 14.1] A' is of harmonic measure 0. This implies a contradiction. Thus f is constant.

COROLLARY. *Let f be a Fatou mapping and let a set $A \subset \Delta$ be of harmonic measure positive. If each point of $A \cap \Delta_1$ is an accumulating point of $\{\lambda_n\}$ with respect to the fine topology and $\{f(\lambda_n)\}$ converges to a point $c \in R'^*$, then f is constant.*

PROOF. Take A' in the proof of the theorem. For any $q \in A'$ and each $D_n \in F_q$ in Lemma 2, take a point $Q_n \in D_n \cap \lambda_N$ for some N; then $\{Q_n\}$ converges to q and $\lim_n f(Q_n) = \hat{f}(q)$. We can choose a subsequence $\{a_n\}$ of $\{Q_n\}$ for which $\{f(a_n)\}$ converges to c. Thus as in the proof of the theorem, we have the assertion of the corollary.

REFERENCES

1. C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer-Verlag, Berlin, 1963. MR **28** #3151.

2. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Univ. Press, New York, 1966. MR **38** #325.

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