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OSCILLATION OF A FORCED SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. Sufficient conditions are given which insure that every solution of $(a(t)y')' + p(t)f(y)g(y') = r(t)$ has arbitrarily large zeros. We seem to have a partial answer to a question posed by Kartsatos [4]. An example is given illustrating the result.

Recently Keener [5], Skidmore and Bowers [8], Skidmore and Leighton [9], Rankin [7] and Tefteller [10] have investigated the oscillation and nonoscillation of solutions of the linear equations $(a(t)v')' + p(t)v(t) = r(t)$. In each of the above papers the results have depended on the associated homogeneous equation $(a(t)u')' + p(t)u(t) = 0$. Thus, these results are restrictive in the sense that the methods are not extendible to the nonlinear case.

Here we will study the equation

$$(a(t)y')' + p(t)f(y)g(y') = r(t). \tag{1}$$

Graef and Spikes [1] have recently given conditions which insure that every strictly positive solution or strictly negative solution or solutions that have arbitrarily large zeros but are of one sign approach zero as t approaches infinity. Their result extends a result of Hammett [2] for equation (1) when $g(y') \equiv 1$. Others who have investigated the behavior of solutions of (1) when $g(y') \equiv 1$ are Kartsatos [3], [4] and Londen [6]; Teufel [11] has investigated the case for $p(t)f(y)g(y') = h(t,y,y')$.

Our result gives sufficient conditions which guarantee that every solution of equation (1) is oscillatory. As usual, a solution $y(t)$ of (1) is oscillatory if for each real number a, there exists a $b > a$ such that $y(b) = 0$. Obviously, we are assuming that all solutions can be extended to infinity.

The following assumptions are made: $a(t)$, $p(t)$, and $r(t)$ are continuous on $[0,\infty)$; $f(x)$ and $g(x)$ are continuous on $(-\infty,\infty)$; $a(t) > 0$ and $p(t) > 0$ on $[0,\infty)$ and

(i) $\int_0^\infty ds/a(s) = \infty$;
(ii) $\int_0^\infty p(s)\,ds = \infty$;
(iii) $xf(x) > 0$ for $x \neq 0$, $df/dx \geqslant 0$ for all x;
(iv) $g(x') \geqslant k > 0$ for all x';

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(v) $|\int_b^t (\int_b^s dw/a(w))r(s)\,ds| < M$ for all $t \geqslant b$ and all $b > 0$.

(vi) There exists an increasing sequence $\{b_n\}$ with $\lim_{n\to\infty} b_n = \infty$ such that for each n,

$$\lim_{t\to\infty} \int_{b_n}^t \left(\int_s^t dw/a(w)\right) r(s)\,ds = L_n.$$

The numbers L_n take on both positive, negative or possibly zero values for arbitrarily large n.

THEOREM 1. *If conditions* (iv)–(vi) *are satisfied, then every solution* $y(t)$ *of equation* (1) *is oscillatory.*

PROOF. First, we notice that for each n,

$$\int_{b_n}^t \left(\int_s^t dw/a(w)\right) r(s)\,ds$$
$$= \left(\int_{b_n}^t dw/a(w)\right) \int_{b_n}^t r(s)\,ds \;-\; \int_{b_n}^t \left(\int_{b_n}^s dw/a(w)\right) r(s)\,ds$$

and so by (i), (v), and (vi),

$$\lim_{t\to\infty} \int_{b_n}^t r(s)\,ds = 0 \quad \text{for all } n. \tag{2}$$

Now assume that there exists a solution $y(t)$ of equation (1) such that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 > 0$. We have by integration of (1) from b_n to t for some $b_n > T_0$ that

$$a(t)y'(t) + \int_{b_n}^t p(s)f(y)g(y')\,ds = a(b_n)y'(b_n) + \int_{b_n}^t r(s)\,ds.$$

From (2) we have

$$\lim_{t\to\infty}\left(a(t)y'(t) + \int_{b_n}^t p(s)f(y)g(y')\,ds\right) = a(b_n)y'(b_n). \tag{3}$$

Since $\int_{b_n}^t p(s)f(y)g(y')\,ds$ is nondecreasing with t, we see that $\lim_{t\to\infty} \int_{b_n}^t p(s)f(y)g(y')\,ds = \alpha$ where $\alpha = \infty$ or α is positive and finite. If $\alpha = \infty$, then $a(t)y'(t)$ must approach $-\infty$ as t approaches ∞ and this with (i) would imply that $y(t)$ is eventually negative. Therefore, α is finite.

Using (2) again we have that $\lim_{t\to\infty} a(t)y'(t) = \beta$. The number β must be zero, for if $\beta > 0$ then $a(t)y'(t) \geqslant c > 0$ for all $t \geqslant T_1 > T_0$ for some T_1. From (i) it would then be true that $\lim_{t\to\infty} y(t) = \infty$. Thus, by (iii) and (iv) there exists an $L > 0$ such that $f(y)g(y') > L$ for all $t \geqslant T_2 > T_1$ for some T_2. Integrating (1) from T_2 to t we get

$$a(t)y'(t) - a(T_2)y'(T_2) + L\int_{T_2}^t p(s)\,ds \leqslant \int_{T_2}^t r(s)\,ds. \tag{4}$$

The right side of (4) is bounded as t goes to infinity while the left side goes to ∞; hence, β is not positive.

If $\beta < 0$, then there would exist an $s > 0$ such that $a(t)y'(t) < -s$ for all

$t \geqslant T_3 > T_0$ for some T_3. This again leads to $y(t)$ becoming negative. Therefore, we must have $\lim_{t\to\infty} a(t)y'(t) = 0$, and from (3),

$$\lim_{t\to\infty} \int_{b_n}^{t} p(s)f(y)g(y')\,ds = a(b_n)y'(b_n). \tag{5}$$

From (5) and since $p(t)f(y(t))q(y'(t)) > 0$ for $t \geqslant b_n$, we have that for each $b_n > T_0$

$$\int_{b_n}^{t} p(s)f(y)g(y')\,ds \leqslant a(b_n)y'(b_n) \quad \text{for all } t \geqslant b_n.$$

Assume that n is such that $L_n \geqslant 0$ and let the positive number ε be such that $L_n - \varepsilon > 0$ if $L_n > 0$ or $y(b_n) - \varepsilon > 0$ if $L_n = 0$. By integrating (1) twice from b_n to t we obtain

$$\begin{aligned} y(t) &= y(b_n) + \int_{b_n}^{t}\left(\int_s^t \frac{dw}{a(w)}\right) r(s)\,ds + a(b_n)y'(b_n)\int_{b_n}^{t}\frac{dw}{a(w)} \\ &\quad - \int_{b_n}^{t}\left(\int_s^t \frac{dw}{a(w)}\right) p(s)f(y)g(y')\,ds \geqslant y(b_n) + L_n - \varepsilon \\ &\quad + \left(\int_{b_n}^{t}\frac{dw}{a(w)}\right)\left[a(b_n)y'(b_n) - \int_{b_n}^{t} p(s)f(y)g(y')\,ds\right] \\ &\geqslant y(b_n) + L_n - \varepsilon = L' > 0 \quad \text{for all } t \geqslant T_4 \text{ for some } T_4 > T_0. \end{aligned}$$

Integrating (1) once more from T_4 to t we have

$$a(t)y'(t) - a(T_4)y'(T_4) + f(L')k\int_{T_4}^{t} p(s)\,ds \leqslant \int_{T_4}^{t} r(s)\,ds. \tag{6}$$

Since $\lim_{t\to\infty} a(t)y'(t) = 0$, we see from (ii) that the left side of (6) approaches ∞ which contradicts the boundedness of $\int_{T_4}^{t} r(s)\,ds$.

Hence, our assumption that $y(t) > 0$ on $[T_0, \infty)$ must be false (the case $y(t) < 0$ is similar and will, therefore, be omitted), implying that $y(t)$ must be oscillatory.

EXAMPLE. Consider the equation

$$y'' + p(t)f(y)g(y') = e^{-t}\sin t. \tag{7}$$

We have that $|\int_b^t (s-b)e^{-s}\sin s\,ds|$ is bounded for all b and all $t \geqslant b$ and that

$$\lim_{t\to\infty}\int_{b_n}^{t}(t-s)e^{-s}\sin s\,ds = L_n = \begin{cases} \sqrt{2}\,e^{-b_n}/2 & \text{if } n \text{ even}, \\ -\sqrt{2}\,e^{-b_n}/2 & \text{if } n \text{ odd}, \end{cases}$$

where $b_n = 3\pi/4 + n\pi$, $n = 0, 1, 2, \ldots$. Thus if conditions (ii)–(iv) are satisfied then every solution of (7) is oscillatory. We can see that condition (vi) is critical by considering the equation

$$y'' + y = e^{-t}. \tag{8}$$

Note that (i)–(v) are satisfied by (8) but (vi) is not. Equation (8) has the nonoscillatory solution $y(t) = e^{-t}/2$.

REMARKS. From our example, we see that our results handle functions $r(t)$ which are not considered in [11]. Thus, along with the results of [11] we seem to have at least a partial answer to a question posed by Kartsatos [4].

We note that for the example considered, the distance between b_n and b_{n+1} remained constant for all n and that the L_n's alternated between strictly positive and negative values. These facts, however, were not needed in the proof of the theorem. In fact, the case $L_n = 0$ for all n would be acceptable. If $r(t) \equiv 0$, we would have this situation and therefore, it seems that our result is a natural extension of what is known for the equation $(a(t)u')' + p(t)f(u)g(u') = 0$.

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