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ON LEBESGUE SUMMABILITY FOR DOUBLE SERIES

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ABSTRACT. In [2] a two dimensional analogue of Lebesgue's theorem on differentiation of formally integrated trigonometric series was established. Here we show that a stronger analogue holds.

1. Let T: $\sum_{n\in\mathbf{Z}} c_n e^{in\theta}$ be a trigonometric series in one variable, with $c_n \to 0$. Let

$$\lambda(\theta) = c_0\theta + \sum_{n\neq 0} \frac{c_n}{in} e^{in\theta}.$$

We say T is *Lebesgue summable* at θ_0 to sum s if $\lambda(\theta)$ has at θ_0 a first symmetric derivative with value s. That is, if $\frac{1}{2}\{\lambda(\theta_0 + t) - \lambda(\theta_0 - t)\} = st + o(t)$ as $t \to 0$. The following result is well known (see [3, p. 322]).

THEOREM A. *Suppose $c_n = O(1/n)$ as $n \to \infty$ and suppose T converges at θ_0 to finite sum s. Then T is Lebesgue summable at θ_0 to s.*

2. We are concerned here with a two dimensional analogue of Theorem A for spherically convergent series. We denote points of T_2 by $x = (x_1, x_2) = te^{i\theta}$ and integral lattice points by $n = (n_1, n_2)$. We write $n \cdot x = n_1x_1 + n_2x_2$ and $|n| = \sqrt{n \cdot n}$.

Let $L(x)$ be defined in a neighborhood of $x_0 \in T_2$. We will say, see [2], that L has at x_0 a *generalized first symmetric derivative* with value s if $L(x)$ is integrable over each circle $|x - x_0| = t$, for t small, and if

$$\frac{1}{2\pi}\int_0^{2\pi} L(x_0 + te^{i\theta})(\cos\theta + \sin\theta)\,d\theta = \tfrac{1}{2}st + o(t) \tag{2.1}$$

as $t \to 0$.

If the limit in (2.1) exists only as t tends to 0 through a set having 0 as a point of density, we will say $L(x)$ has at x_0 a *generalized first symmetric approximate derivative.*

The following result was established in [2].

THEOREM B. *Let T: $\sum_{n\in\mathbf{Z}_2} c_n e^{in\cdot x}$ be a double trigonometric series which converges spherically at x_0 to s, $|s| < \infty$. Suppose the coefficients of T satisfy*

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$$(2.2) \qquad \sum_{n_1+n_2=0} |n|^\alpha |c_n|^2 + \sum_{n_1+n_2\neq 0} |n|^\alpha (n_1+n_2)^{-2} |c_n|^2 < \infty$$

for some number $\alpha > 1$. *Then the series*

$$(2.3) \qquad \sum_{n_1+n_2=0} \frac{1}{2}(x_1+x_2) c_n e^{in\cdot x} + \sum_{n_1+n_2\neq 0} \frac{-ic_n}{n_1+n_2} e^{in\cdot x}$$

converges spherically a.e. on T_2 *to a function* $L(x)$ *which has at* x_0 *a generalized first symmetric approximate derivative equal to* s.

3. In this paper we improve Theorem B by eliminating the word "approximate" from its conclusion. Thus we attain a closer analogue to Theorem A. Our result is

THEOREM C. *Let* T: $\sum_{n\in \mathbf{Z}_2} c_n e^{in\cdot x}$ *be a double trigonometric series which converges spherically at* x_0 *to* s, $|s| < \infty$. *Suppose the coefficients of* T *satisfy* (2.2) *for some number* $\alpha > 1$. *Then the series* (2.3) *converges spherically a.e. on* T_2 *to a function* $L(x)$ *which has at* x_0 *a generalized first symmetric derivative with value* s.

4. PROOF OF THEOREM C. We may assume $x_0 = 0$, and $c_0 = s = 0$. Write $S_u = S_u(0) = \sum_{|n|<u} c_n$. Let

$$g_n(x) = \begin{cases} \dfrac{-i}{n_1+n_2} e^{in\cdot x} & \text{if } n_1+n_2 \neq 0, \\ \frac{1}{2}(x_1+x_2) e^{in\cdot x} & \text{if } n_1+n_2 = 0. \end{cases}$$

Then, for $n \neq 0$, by the Lemma of [2],

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})(\cos\theta + \sin\theta)\, d\theta = \frac{J_1(|n|t)}{|n|}.$$

Let

$$L_R(x) \sum_{\substack{|n|<R \\ n_1+n_2=0}} \frac{1}{2}(x_1+x_2) c_n e^{in\cdot x} + \sum_{\substack{|n|<R \\ n_1+n_2\neq 0}} \frac{-ic_n}{n_1+n_2} e^{in\cdot x}.$$

The condition (2.2) on c_n insures that $L(x) = \lim_{R\to\infty} L_R(x)$ exists a.e. on each circle $|x| = t$. This is a consequence of Theorem 1 of [1]. Moreover, by Theorem 2 of [1],

$$\int_0^{2\pi} \sup_R |L_R(te^{i\theta})|\, d\theta < \infty,$$

so we may integrate the series defining $L(x)$ term by term over each circle. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \lim_{R\to\infty} \sum_{|n|<R} c_n g_n(te^{i\theta})(\cos\theta+\sin\theta)\,d\theta$$

$$(4.1)\qquad = \lim_{R\to\infty} \sum_{|n|<R} c_n \frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})(\cos\theta+\sin\theta)\,d\theta$$

$$= \lim_{R\to\infty} \sum_{|n|<R} c_n |n|^{-1} J_1(|n|t)$$

$$= \lim_{R\to\infty} t \sum_{|n|<R} c_n \gamma(|n|t),$$

where $\gamma(z) = z^{-1}J_1(z)$.

We express the last sum as an integral and integrate by parts.

$$\sum_{|n|<R} c_n\gamma(|n|t) = S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)\,du.$$

Since the series T is spherically convergent to 0 at $x_0 = 0$ and since $J_1(z) = O(z^{-1/2})$ as $z \to \infty$,

$$S_R\gamma(Rt) = o(1)O(R^{-3/2}) = o(1)$$

as $R \to \infty$. Therefore, returning to (4.1)

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta+\sin\theta)\,d\theta$$

$$= \lim_{R\to\infty} t \sum_{|n|<R} c_n \gamma(|n|t)$$

$$= -t\int_0^{\infty} S_u \frac{d}{du}\gamma(ut)\,du$$

$$= -t\left\{\int_0^{1/t} S_u \frac{d}{du}\gamma(ut)\,du + \int_{1/t}^{\infty} S_u \frac{d}{du}\gamma(ut)\,du\right\}$$

$$= -t\{A(t)+B(t)\}.$$

We will show $A(t)$ and $B(t)$ each tend to 0 with t. To estimate $A(t)$ we note that $\gamma(z)$ is an entire function, so for $|z| < 1$, $|\gamma'(z)| \leqslant K$.

$$A(t) = \int_0^{1/t} o(1)\cdot tK\,du = o(1).$$

To estimate B we note

$$\frac{d}{dz}\gamma(z) = \frac{d}{dz}(z^{-1}J_1(z)) = -z^{-1}J_2(z) = z^{-1}O(z^{-1/2}) = O(z^{-3/2}).$$

Thus

$$B(t) = \int_{1/t}^{\infty} o(1) \cdot tO(ut)^{-3/2}\,du = o(1).$$

This completes the proof of Theorem C.

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