ON LEBESGUE SUMMABILITY FOR DOUBLE SERIES

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Abstract. In [2] a two dimensional analogue of Lebesgue’s theorem on
differentiation of formally integrated trigonometric series was established.
Here we show that a stronger analogue holds.

1. Let \( T: \sum_{n \in \mathbb{Z}^2} c_n e^{in\theta} \) be a trigonometric series in one variable, with \( c_n \to 0 \).
Let
\[
\lambda(\theta) = c_0 \theta + \sum_{n \neq 0} \frac{c_n}{in} e^{in\theta}.
\]
We say \( T \) is Lebesgue summable at \( \theta_0 \) to sum \( s \) if \( \lambda(\theta) \) has at \( \theta_0 \) a first symmetric
derivative with value \( s \). That is, if \( \frac{1}{2} \{ \lambda(\theta_0 + t) - \lambda(\theta_0 - t) \} = st + o(t) \) as \( t \to 0 \). The following result is well known (see [3, p. 322]).

Theorem A. Suppose \( c_n = O(1/n) \) as \( n \to \infty \) and suppose \( T \) converges at \( \theta_0 \)
to finite sum \( s \). Then \( T \) is Lebesgue summable at \( \theta_0 \) to \( s \).

2. We are concerned here with a two dimensional analogue of Theorem A
for spherically convergent series. We denote points of \( T_2 \) by \( x = (x_1, x_2) \)
= \( te^{i\theta} \) and integral lattice points by \( n = (n_1, n_2) \). We write \( n \cdot x = n_1 x_1 + n_2 x_2 \) and \( |n| = \sqrt{n \cdot n} \).
Let \( L(x) \) be defined in a neighborhood of \( x_0 \in T_2 \). We will say, see [2], that
\( L(x) \) has at \( x_0 \) a generalized first symmetric derivative with value \( s \) if \( L(x) \) is
integrable over each circle \( |x - x_0| = t \), for \( t \) small, and if
\[
(2.1) \quad \frac{1}{2\pi} \int_0^{2\pi} L(x_0 + te^{i\theta}) (\cos \theta + \sin \theta) d\theta = \frac{1}{2} st + o(t)
\]
as \( t \to 0 \).
If the limit in (2.1) exists only as \( t \) tends to 0 through a set having 0 as a
point of density, we will say \( L(x) \) has at \( x_0 \) a generalized first symmetric
approximate derivative.
The following result was established in [2].

Theorem B. Let \( T: \sum_{n \in \mathbb{Z}^2} c_n e^{in\theta} \) be a double trigonometric series which
converges spherically at \( x_0 \) to \( s \), \( |s| < \infty \). Suppose the coefficients of \( T \) satisfy
for some number $\alpha > 1$. Then the series

$$\sum_{n_1+n_2=0} \frac{1}{2} (x_1 + x_2) c_n e^{i n\cdot x} + \sum_{n_1+n_2 \neq 0} -\frac{i c_n}{n_1 + n_2} e^{i n\cdot x}$$

converges spherically a.e. on $T_2$ to a function $L(x)$ which has at $x_0$ a generalized first symmetric approximate derivative equal to $s$.

3. In this paper we improve Theorem B by eliminating the word "approximate" from its conclusion. Thus we attain a closer analogue to Theorem A. Our result is

**Theorem C.** Let $T: \sum_{n \in \mathbb{Z}^2} c_n e^{i n\cdot x}$ be a double trigonometric series which converges spherically at $x_0$ to $s$, $|s| < \infty$. Suppose the coefficients of $T$ satisfy (2.2) for some number $\alpha > 1$. Then the series (2.3) converges spherically a.e. on $T_2$ to a function $L(x)$ which has at $x_0$ a generalized first symmetric derivative with value $s$.

4. **Proof of Theorem C.** We may assume $x_0 = 0$, and $c_0 = s = 0$. Write $S_u = S_u(0) = \sum_{|n|<u} c_n$. Let

$$g_n(x) = \begin{cases} \frac{-i}{n_1 + n_2} e^{i n\cdot x} & \text{if } n_1 + n_2 \neq 0, \\ \frac{1}{2} (x_1 + x_2) e^{i n\cdot x} & \text{if } n_1 + n_2 = 0. \end{cases}$$

Then, for $n \neq 0$, by the Lemma of [2],

$$\frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta})(\cos \theta + \sin \theta) d\theta = J_f(|n|) \frac{1}{|n|}.$$

Let

$$L_R(x) = \sum_{|n|<R \atop n_1+n_2=0} \frac{1}{2} (x_1 + x_2) c_n e^{i n\cdot x} + \sum_{|n|<R \atop n_1+n_2 \neq 0} -\frac{i c_n}{n_1 + n_2} e^{i n\cdot x}.$$

The condition (2.2) on $c_n$ insures that $L(x) = \lim_{R \to \infty} L_R(x)$ exists a.e. on each circle $|x| = t$. This is a consequence of Theorem 1 of [1]. Moreover, by Theorem 2 of [1],

$$\int_0^{2\pi} \sup_R |L_R(te^{i\theta})| d\theta < \infty,$$

so we may integrate the series defining $L(x)$ term by term over each circle. Thus,
where $\gamma(z) = z^{-1}J_1(z)$.

We express the last sum as an integral and integrate by parts.

$$\sum_{|n|<R} c_n \gamma(|n|) = S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(u) du.$$ 

Since the series $T$ is spherically convergent to 0 at $x_0 = 0$ and since $J_1(z) = O(z^{-1/2})$ as $z \to \infty$,

$$S_R \gamma(Rt) = o(1)O(R^{-3/2}) = o(1)$$ 

as $R \to \infty$. Therefore, returning to (4.1)

$$\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta$$

$$= \lim_{R \to \infty} \sum_{|n|<R} c_n \gamma(|n|)$$

$$= -t \int_0^\infty S_u \frac{d}{du} \gamma(u) du$$

$$= -t \left\{ \int_0^{1/t} S_u \frac{d}{du} \gamma(u) du + \int_{1/t}^\infty S_u \frac{d}{du} \gamma(u) du \right\}$$

$$= -t \{ A(t) + B(t) \}.$$

We will show $A(t)$ and $B(t)$ each tend to 0 with $t$. To estimate $A(t)$ we note that $\gamma(z)$ is an entire function, so for $|z| < 1$, $|\gamma'(z)| \leq K$.

$$A(t) = \int_0^{1/t} o(1) \cdot tK du = o(1).$$

To estimate $B$ we note

$$\frac{d}{dz} \gamma(z) = \frac{d}{dz} (z^{-1}J_1(z)) = -z^{-1}J_2(z) = z^{-1}O(z^{-1/2}) = O(z^{-3/2}).$$

Thus
\[ B(t) = \int_{1/t}^{\infty} o(1) \cdot tO(ut)^{-3/2} \, du = o(1). \]

This completes the proof of Theorem C.

**Bibliography**


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