GROUP RINGS WITH SOLVABLE $n$-ENGLl UNIT GROUPS$^1$

J. L. FISHER, M. M. PARMENTER AND S. K. SEHGAL

ABSTRACT. Let $KG$ be the group ring of a group $G$ over a field of characteristic $p > 0$, $p \neq 2, 3$. Suppose $G$ contains no element of order $p$ (if $p > 0$). Group algebras $KG$ with unit group $U(KG)$ solvable and $n$-Engel are characterized.

Let $KG$ be the group ring of a group $G$ over a field $K$ of characteristic $p > 0$ and let $U(KG)$ denote its group of units. Several authors including Bateman [1], Bateman and Coleman [2], Motose and Tominaga [10] and Khripta [5] have studied the question as to when $U(KG)$ is solvable or nilpotent. Khripta in a beautiful paper [5] has proved that if $p > 0$ and $G$ has a $p$-element then $U(KG)$ is nilpotent if and only if $G$ is nilpotent and the derived group $G'$ is a finite $p$-group, settling the nonsemiprime case. This, incidently, is equivalent to saying that $KG$ is Lie nilpotent (see [11] and [14]). Khripta also has some results in her thesis on the nilpotency of $U(KG)$ in the semiprime case. We investigate when $U(KG)$ is a solvable $n$-Engel group; more precisely we prove

**Theorem.** Suppose $KG$ is a group ring over a field $K$ of characteristic $p > 0$, $p \neq 2, 3$. Suppose $G$ has no element of order $p$ (if $p > 0$). Then the following are equivalent.

(i) $U(KG)$ is solvable and $n$-Engel.

(ii) $G$ is solvable and $m$-Engel and one of (a), (b) holds.

(a) $T(G)$, the set of torsion elements of $G$, is central in $G$.

(b) $|K| = 2^p - 1 = p$, a Mersenne prime; $T(G)$ is abelian of exponent $(p^2 - 1)$ and for $x \in G$, $t \in T(G)$, $xt \neq tx \Rightarrow x^{-1}tx = t^p$.

(iii) $U(KG)$ is nilpotent.

We are indebted to the referee for several useful comments.

1. Notations and definitions. For group elements $x, y$ we write the commutator $(x, y) = xyx^{-1}y^{-1}$ and

\[ (x, y, \underbrace{y, \ldots, y}_{n+1}) = (x, y, \underbrace{y, \ldots, y}_{n})y(x, \underbrace{y, \ldots, y}_{n})^{-1}y^{-1}. \]

A group $H$ is $n$-Engel if it satisfies

\[ (x, y, \ldots, y) = 1 \quad \text{for all } x, y \in H \]

$^1$This work has been supported by N.R.C. Grant Nos. A-5300, A-8775 and A-7550.
and fixed \( n \). Let \( F \) be the multiplicative group of a field \( F \). We denote by \( \mathcal{E} = \mathcal{E}(\hat{F}) \), the ring of endomorphisms of \( \hat{F} \). We write \( f^\alpha \) for the image of \( f \) under \( \alpha \) for \( f \in \hat{F} \), \( \alpha \in \mathcal{E} \). Thus \( f^{\alpha+\beta} = f^\alpha \cdot f^\beta \) and \( f^{\alpha \beta} = (f^\alpha)^\beta \).

By a crossed product \( K(G, \rho_{g,h}, \alpha_g) \), we understand the set of finite sums, \( \{ \sum k_i g_i | k_i \in K, g_i \in G \} \) where \( g_i \) is a symbol corresponding to \( g_i \) and \( \rho: G \times G \to K \) is a factor system and \( \alpha_g \) is an automorphism of \( K \) for each \( g \in G \). Equality and addition are defined componentwise. And, for \( g, h \in G \), \( k \in K, \ g \cdot h = \rho_{g,h} g h, \ g k = k^{\alpha_g} h \) where \( \rho \) and \( \alpha \) are required to satisfy the necessary conditions for \( K(G, \rho_{g,h}, \alpha_g) \) to be a ring. For details, we refer to [3].

As a special case, if we have \( \alpha_g = I \) for all \( g \in G \), we call \( K(G, \rho_{g,h}, I) = K^I(G) \) the twisted group ring (see [12]). If \( \rho_{g,h} = 1 \) for all \( g, h \in G \), we call \( K(G, 1, \alpha_g) \), the skew group ring and denote it by \( K_\alpha(G) \). And, of course, if also \( \alpha_g = I \) for all \( g \in G \), we have the (ordinary) group ring. We shall have occasion to use both skew and twisted group rings.

2. The skew group ring of an infinite cyclic group. Let \( F \) be a field contained in \( KG \). Suppose that \( x \in G \) has infinite order, \( \langle x \rangle \) is linearly independent over \( F \), and that \( x \) induces an automorphism \( \alpha = \alpha_x \) of \( F \) by conjugation, i.e. \( \alpha: f \to x f x^{-1} = f^x \). Then we have an isomorphic copy of the skew group ring \( F_x \langle x \rangle \) contained in \( KG \). Hence \( F_x \langle x \rangle = \{ \sum f_i x^i | f_i \in F \} \) where addition and equality are componentwise and \( x f = f^x \). We investigate \( F_x \langle x \rangle \) in this section.

**Lemma 2.1.** For all \( f \in \hat{F} \), we have

\[
(f, x, x, \ldots, x) = f^{(1-\alpha)^m}.
\]

**Proof.** We use induction on \( m \). Notice that

\[
(f, x) = f x f^{-1} x^{-1} = f \cdot (f^{-1})^\alpha = f^{-\alpha} = f^{(1-\alpha)}.
\]

Suppose we already know that (2.2) holds for \( m \); then

\[
(f, x, x, \ldots, x) = f^{(1-\alpha)^m} x (f^{(1-\alpha)^m})^{-1} x^{-1} = f^{(1-\alpha)^m} x f^{-(1-\alpha)^m} x^{-1} = f^{(1-\alpha)^{m+1}}.
\]

The lemma is proved.

**Proposition 2.3.** Let \( F \) be an infinite field of characteristic \( p \geq 0 \) and \( \alpha \) be an automorphism of finite order. Suppose that in \( F_x \langle x \rangle \) we have

\[
(f, x, x, \ldots, x) = 1 \quad \text{for all } f \in \hat{F}.
\]

Then \( F_x \langle x \rangle = F \langle x \rangle \), i.e. \( \alpha \) is the identity automorphism.

**Proof.** We have by the last lemma, for all \( f \in \hat{F} \),
Let $s > 1$ be the order of $\alpha$. Choose a prime $q > \max(p, m)$ of the form $2sk + 1$. Then
\[ 1 = f^{(1-\alpha)^m} = f^{\Sigma^{(1-\alpha)^m}(-1)^{\ell}(\binom{m}{\ell})\alpha^\ell}. \]

The finite automorphism group $I, \alpha, \alpha^2, \ldots, \alpha^{s-1}$ satisfies
\[ h(f, \alpha(f), \ldots, \alpha^{s-1}(f)) = 0 \]
with
\[ h(X_0, X_1, \ldots, X_{s-1}) = X_0(\alpha^0 \cdot X_1^\alpha \cdot X_2^{\alpha^2} \cdots X_{s-1}^{\alpha^{s-1}} - X_1^{b_1} \cdot X_2^{b_2} \cdots X_{s-1}^{b_{s-1}}), \]
and $a = |\Sigma^{2k}(-1)^\ell\binom{m}{\ell}|$. Since $a \equiv 1 \mod q$, (2.4) is a nontrivial polynomial, contradicting Artin's theorem on the algebraic independence of automorphisms of an infinite field [6, p. 228]. Hence $s = 1$ and $a = I$.

**Proposition 2.5.** Let $F$ be a finite field of $p^a$ elements. If $F_a(x)$ satisfies
\[ (f, x, x, \ldots, x) = 1 \quad \text{for all } f \in \hat{F}, \]
and $\alpha$ is not the identity automorphism; then, $f^a = f^p$ for all $f \in \hat{F}$ and $|F| = p^2$, where $p$ is a Mersenne prime.

**Proof.** Since $f^a = f^p$ for some $j < a$, we have that
\[ (f, x, x, \ldots, x) = f^{(1-\alpha)^m} = f^{(1-p^j)^m} = 1 \quad \text{for all } f \in \hat{F}. \]
Therefore, $(p^a - 1)$ divides $(p^j - 1)^m$. Hence,
\[ (2.6) \quad \text{any prime divisor of } (p^a - 1) \text{ divides } (p^j - 1). \]

We claim that (2.6) implies $a = 2$. Let $j$ be the smallest natural number such that (2.6) holds for a fixed $a$. Then writing, $a = jq + r$,
\[ p^a - 1 = p^{jq+r} - 1 = p^r(p^{jq} - 1) + (p^r - 1), \]
it follows that any prime divisor of $(p^a - 1)$ is a divisor of $(p^j - 1)$. We may thus assume that $a = jq$. We have now that any prime divisor of $(p^j)^q - 1$ is a divisor of $(p^j - 1)$. It is easy to see (cf. [9]) that $q = 2$ and $p^j = 2^\gamma - 1$. It follows by [15, p. 335] that $j = 1$. Thus $a = 2$. We have therefore proved that $|F| = p^2, p = 2^\gamma - 1$ and hence $f^a = f^p$.

3. **Proof of the theorem.** We need the following crucial result of Lanski.

**Theorem 3.1 (Lanski).** Let $R$ be a semiprime ring which is 6-torsion free. If $U(R)$ is solvable, then all idempotents of $R$ are central.

**Proof.** See [7, Lemma 5] and [8, Theorem 9 and §1].
We shall prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

3.2 (i) $\Rightarrow$ (ii): Let $g$ and $h$ be elements of finite order of $G$. Since

$$e = (1/O(g)) \sum_{i} g^i$$

is an idempotent, $\langle g \rangle$ is normal by (3.1). Also $\langle h \rangle$ is normal. Thus $T_0 = \langle g, h \rangle$ is a finite normal subgroup of $G$. Now,

$$K_T = \sum_{i} (D_i)_{n_i}$$

is a direct sum of full matrix rings $(D_i)_{n_i}$ over division rings $D_i$. It follows by [4] that each $n_i = 1$ and each $D_i$ is a commutative field $F_i$. Hence $gh = hg$. Thus $T = T(G)$, the torsion elements of $G$ form a normal abelian subgroup of $G$.

Let $x \in G$, $x \not\in T$ and let $T_0$ be a finite subgroup of $T$. Suppose that $x$ does not commute with $T_0$ elementwise. Since every finite subgroup is normal in $G$, the skew group ring $(K_T)_{a\langle x \rangle}$ is contained in $K_T$, where $a$ is the automorphism of $KT_0$ induced by conjugation by $x$. Now, $K_T = \sum_{i} F_i$, where $F_i$ are fields. Also,

$$(3.3) K_T \supset (K_T)_{a\langle x \rangle} = \left( \sum_{i} F_i \right)_{a\langle x \rangle} \simeq \sum_{i} (F_i)_{a\langle x \rangle}.$$ 

The last isomorphism follows because every idempotent is central in $K_T$ by (3.1) and $xF_ix^{-1} = F_i$.

We can conclude from (3.3) that the unit group of each $(F_i)_{a\langle x \rangle}$ is $n$-Engel. Since $F_i$ is algebraic over $K$, it follows by Propositions 2.3 and 2.5 that $|K| = p$ or $p^2$, where $p$ is a Mersenne prime. If $|K| = p^2$ then

$$|F_i| = |K| \Rightarrow F_i = e(KT_0) = eK, \quad e^2 = e.$$ 

Since every idempotent is central, $F_i$ commutes with $x$. Thus we have $|K| = p = 2^p - 1$. It remains to prove that $T_0(p^2 - 1) = 1$ and

$$xt \neq tx, \quad t \in T_0 \Rightarrow x^{-1}tx = t^p.$$ 

We first make two observations. Write $T_0 = E \times A$, where $E$ is a 2-group and $A$ is an odd group.

3.4. $A$ is central.

Let $g \in A$, then since $x^2$ is central, $\langle x, g \rangle / \langle x^2 \rangle$ is a nilpotent group of order $2 \cdot O(g)$. Thus $xgx^{-1} = gx^{2i}$ and also $xgx^{-1} = g^i$ as $\langle g \rangle$ is normal in $G$. Hence $xgx^{-1} = g$.

3.5. If $g$ and $h$ are nonidentity elements of $T_0$ then $(1 - g)(1 - h) \neq 0$. This is because the coefficient of identity in this product is 1 or 2 and $p \neq 2$.

Suppose that $T_0(p^2 - 1) \neq 1$. Choose $g, h \in T_0$ with $h^xh^{-1} \neq 1$ and $g^{p-2} \neq 1$. Then

$$\pi = (1 - g^{p-2})(1 - h^xh^{-1}) \neq 0.$$ 

Therefore, there exists an $F_i$ and a homomorphism

$$\lambda: KT_0 \rightarrow F_i$$
with \( \lambda(\pi) \neq 0 \). Thus \( \lambda(\sigma) = \lambda(h)^{p^2-1} \neq 1 \) and \( |F_i| > p^2 \). Since \( \lambda(h^xh^{-1}) \neq 1 \), we have \( \lambda(\sigma) = \lambda(h)^{x} \neq \lambda(h) \) and \( F_i \) is not central, contradicting Proposition 2.5. We have therefore proved that \( T_0^{(p^2-1)} = 1 \).

In order to complete the proof of the implication (i) \( \Rightarrow \) (ii) it suffices to prove

(3.7) \( g \in T_0, \quad xg \neq gx \Rightarrow x^{-1}gx = g^p \).

We can write \( g = g_1g_2, \quad O(g_1) = 2^r \) and \( O(g_2) \) a divisor of \((p - 1)/2\). Since \( g_2^p = g_2 \) and \( g_2 \) is central due to (3.4), we have only to prove that \( x^{-1}g_1x = g_1^p \).

We may assume that \( s \geq 1 \). Suppose that \( K\langle g_1 \rangle = F_1 \oplus F_2 \oplus \cdots \), \( |F_1| = p^2 = |F_2| \) and \( g_1 = (\xi, \eta, \ldots) \), \( x^{-1}g_1x = (\xi^p, \eta, \ldots) \). Since \( x^{-1}g_1x = g_1 \), we have \( p - i \equiv 0 \pmod{4} \) and \( i - 1 \equiv 0 \pmod{4} \) and thus \( p - 1 \equiv 0 \pmod{4} \) which is a contradiction. Hence \( x^{-1}gx = g^p \).

3.8. (ii) \( \Rightarrow \) (iii).

3.9. We assert that every idempotent of \( KT \) is central in \( KG \). If (ii)(a) holds, the assertion is trivial. So let us assume (ii)(b). Let \( e = e^p = \sum e_ig^p \) and therefore \( e_\alpha = e_\alpha^p \). Now \( e^x = \sum e_\alpha g^x = e \), since \( g^x = g \) or \( g^p \).

Since \( G \) is \( m \)-Engel solvable it follows by [13, Theorem 7.36] that \( G/T(G) \) is nilpotent (say of class \( c \)). We have that either \( T(G) \) is central or \( |K| = p = 2^{\beta} - 1 \) satisfying (ii)(b). We shall prove that \( U(KG) \) is nilpotent of class \( c + \beta + 1 \). We may therefore assume that \( G \) is finitely generated and, hence, by [13, Theorem 7.34] that \( G \) is nilpotent. Therefore \( T = T(G) \) is finite.

We have, \( KT = \bigoplus F_i \) a finite direct sum of fields. Due to (3.9),

\[
KG = (KT)(G/T, \rho, \alpha) = \bigoplus F_i(G/T, \rho, \alpha).
\]

Since \( G/T \) is ordered, \( U(KG) = \bigcap F_i \cdot G/T \). It suffices to prove that \( F_i \cdot G/T \) is nilpotent of class \( \leq c + \beta + 1 \). This is clear if \( \alpha \) is trivial, i.e. if \( F_i \) and \( G/T \) commute. We may therefore suppose that we have \( |F_i| = p^2 \), \( p = 2^\beta - 1 \) and we wish to prove that \( F_i \cdot G/T \) is nilpotent of class \( < c + \beta + 1 \). It is easy to see that \( F_i \subseteq z_{\beta + 1} \), the \( (\beta + 1) \)th term of the upper central series of \( (F_i \cdot G/T) \). Since \( G/T \) is nilpotent of class \( \leq c \), \( F_i \cdot G/T \) is nilpotent of class \( < (c + \beta + 1) \).

3.10. (iii) \( \Rightarrow \) (i) is trivial.

References


Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada
(Current address of J. L. Fisher and S. K. Sehgal)

Department of Mathematics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada (Current address of M. M. Parmenter)