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A CHARACTERISATION OF LIPSCHITZ CLASSES ON FINITE DIMENSIONAL GROUPS

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ABSTRACT. An analogue of a theorem of S. N. Bernstein is developed for certain metric locally compact abelian groups. This, together with a corresponding Jackson-type theorem, gives a characterisation in terms of their Fourier transforms of the Lipschitz functions defined on a compact abelian group with finite topological dimension.

Let G denote a metric locally compact abelian (LCA) group, with translation-invariant metric d, and character group Γ. We shall choose Haar measures λ, θ for G, Γ respectively so that Plancherel's theorem is valid.

It will be necessary to specify metrics for various standard groups, together with their finite products and homomorphic images. The real line $\mathbf{R}$ will be taken with its usual Euclidean metric. For any infinite first countable 0-dimensional LCA group G we take a neighbourhood basis (V_n) at zero consisting of a strictly decreasing sequence of compact open subgroups of G (for the existence of such a basis see [4, (7.7)]), any strictly decreasing sequence (β_n) of positive numbers tending to zero, and define d on $G \times G$ by

$$\begin{aligned} d(x,y) &= \beta_{n+1}, & x-y &\in V_n \backslash V_{n+1}, \\ &= \beta_1, & x-y &\notin V_1, \\ &= 0, & x &= y. \end{aligned}$$

It is easy to verify that d so obtained is a translation-invariant metric on G which generates the given topology. In the particular case when $G = \Delta_{\mathbf{a}}$, the group of $\mathbf{a}$-adic integers, where $\mathbf{a} = (a_0, a_1, \ldots)$ and each a_n is an integer greater than 1, we take the previous metric defined with respect to the basis (Λ_n); here Λ_n is the compact open subgroup of $\Delta_{\mathbf{a}}$ given by

$$\Lambda_n = \{\mathbf{x} \in \Delta_{\mathbf{a}} \colon x_k = 0 \text{ for } k < n\}.$$

Given metric LCA groups (G,d), (G',d'), the product group $G \times G'$ will always be metrised by

$$d''((x,x'),(y,y')) = \max\{d(x,y), d'(x',y')\}.$$

If H is a closed subgroup of G we metrise the quotient group G/H by

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$$d^*(x + H, y + H) = \inf\{d(a,b)\colon a \in x + H, b \in y + H\}.$$

The character group $\Gamma_{G\times G'}$ of a product $G \times G'$ is topologically isomorphic with the product $\Gamma_G \times \Gamma_{G'}$; a typical element $[\gamma, \gamma']$ of $\Gamma_{G\times G'}$ is defined by

$$[\gamma, \gamma']((x, x')) = \gamma(x)\gamma'(x'), \qquad (x, x') \in G \times G',$$

where $\gamma \in \Gamma_G$, $\gamma' \in \Gamma_{G'}$. Given sets $\Upsilon \subset \Gamma_G$, $\Upsilon' \subset \Gamma_{G'}$ we shall write

$$[\Upsilon, \Upsilon'] = \{[\gamma, \gamma']\colon \gamma \in \Upsilon, \gamma' \in \Upsilon'\}.$$

If H is a closed subgroup of G then the character group $\Gamma_{G/H}$ of G/H is topologically isomorphic with $A(\Gamma, H)$ (the annihilator of H in Γ) where, to each $\gamma \in A(\Gamma, H)$, there corresponds $\gamma^+ \in \Gamma_{G/H}$ such that

$$\gamma^+(x + H) = \gamma(x), \qquad x \in G.$$

Given $\Xi \subset A(\Gamma, H)$ we write

$$\Xi^+ = \{\gamma^+ \in \Gamma_{G/H}\colon \gamma \in \Xi\}.$$

We shall also denote by π_H the natural homomorphism of G onto G/H.

The theorems of Jackson and Bernstein (see [6, Chapter 3, Theorems (13.6), (13.20)] respectively) connect the modulus of continuity of a function f with the degree of approximation of f by functions with (certain) compact spectra. The mean modulus of continuity with exponent p of f is given by

$$\omega(p; f; \delta) = \sup\{\|\tau_a f - f\|_p\colon d(a, 0) \leqslant \delta\},$$

where $\tau_a f\colon x \to f(x - a)$. If $f \in L^p(G)$ has the property

$$\omega(p; f; \delta) = O(\delta^\alpha), \qquad \delta > 0,$$

for some $\alpha > 0$, then we say that f is of Lipschitz order α. The functions of Lipschitz order α form a subspace of $L^p(G)$, which we denote by $\mathrm{Lip}_p\,\alpha$.

The spectrum $\Sigma(f)$ is defined as in [4, (40.21)] for $f \in L^\infty(G)$, and by

$$\Sigma(f) = \cup\{\Sigma(\phi * f)\colon \phi \in C_{00}(G)\}$$

(where $C_{00}(G)$ denotes the space of continuous functions on G with compact support) for $f \in L^p(G), p \in [1, \infty)$. When $p = 1$ we find that $\Sigma(f) = \operatorname{supp}(\hat{f})$.

We write

$$L^p_\Upsilon(G) = \{f \in L^p(G)\colon \Sigma(f) \subset \Upsilon\},$$

$$E_\Upsilon(p; f) = \inf\{\|f - t\|_p\colon t \in L^p_\Upsilon(G)\},$$

$$\omega_\Upsilon(a) = \sup\{|\gamma(a) - 1|\colon \gamma \in \Upsilon\}, \qquad a \in G;$$

in the above expressions Υ is a nonvoid subset of Γ.

Our first result is the following analogue of Bernstein's theorem:

THEOREM 1. *Let G be a metric LCA group. Suppose we have an ascending family $\{\Upsilon_n\}_{n=1}^\infty$ of symmetric compact neighbourhoods of zero in Γ, a sequence*

(β_n) of positive numbers smaller than 1, *and positive constants C, K, μ $(\mu < 1)$ such that for each $n \in \{1, 2, \ldots\}$,*

(a) $\omega_{\Upsilon_n}(a) \leqslant C\beta_n^{-1} d(a,0)$, $a \in G$,

(b) $\beta_{n+1} \leqslant \mu\beta_n$,

(c) $\theta(3\Upsilon_n) \leqslant K\theta(\Upsilon_n)$.

Then any $f \in L^p(G)$ with the property that $E_{\Upsilon_n}(p;f) = O(\beta_{n+1}^\alpha)$ for some $\alpha > 0$ satisfies

$$\begin{aligned} \omega(p;f;\delta) &= O(\delta^\alpha), && 0 < \alpha < 1, \\ &= O(\delta|\log\delta|), && \alpha = 1, \\ &= O(\delta), && \alpha > 1. \end{aligned}$$

PROOF. Firstly we note that, with a slight modification of the proof, the lemma in [1] is valid for any LCA group (consider weak*-cluster points rather than weak*-convergent subsequences). This guarantees that for each n there exists $t_n^* \in L_{\Upsilon_n}^p(G)$ for which $E_{\Upsilon_n}(p;f) = \|f - t_n^*\|_p$. By assumption

$$\|f - t_n^*\|_p \leqslant B\beta_{n+1}^\alpha$$

for some constant $B > 0$. Defining

$$s_1 = t_1^*; \quad s_n - t_n^* - t_{n-1}^* \qquad (n \in \{2,3,\ldots\})$$

we have (recall that (β_n) is decreasing)

$$\|s_n\|_p \leqslant \|t_n^* - f\|_p + \|f - t_{n-1}^*\|_p \leqslant 2B\beta_n^\alpha \qquad (n \in \{2,3,\ldots\}).$$

Hence we can find $B' > 0$ such that for all $n \in \{1,2,\ldots\}$

$$\|s_n\|_p \leqslant B'\beta_n^\alpha. \tag{1}$$

Now $\sum_{k=1}^n s_k = t_n^*$ converges in $L^p(G)$ to f as $n \to \infty$. Consequently, for any $a \in G$, $\tau_a(\sum_{k=1}^n s_k) - \sum_{k=1}^n s_k$ converges in $L^p(G)$ to $\tau_a f - f$ as $n \to \infty$, and

$$\|\tau_a f - f\|_p \leqslant \sum_{k=1}^\infty \|\tau_a s_k - s_k\|_p \leqslant \sum_{k=1}^m \|\tau_a s_k - s_k\|_p + 2\sum_{k=m+1}^\infty \|s_k\|_p. \tag{2}$$

The proof of [2, Theorem 1.3] can be adapted to show that

$$\begin{aligned} \|\tau_a s_k - s_k\|_p &\leqslant 4\Big(\theta(3\Upsilon_k)/\theta(\Upsilon_k)\Big)^{1/2} \omega_{\Upsilon_k}(a)\|s_k\|_p \\ &\leqslant 4K^{1/2}C\beta_k^{-1} d(a,0)\|s_k\|_p, \end{aligned} \tag{3}$$

the last step using (a) and (c) above. A combination of (1), (2) and (3) gives

$$\omega(p;f;\delta) \leqslant 4K^{1/2}CB'\delta \sum_{k=1}^m \beta_k^{\alpha-1} + 2B' \sum_{k=m+1}^\infty \beta_k^\alpha$$

for any $\delta > 0$.

Now suppose that $0 < \delta \leqslant \beta_1$, and choose $m \geqslant 1$ so that $\beta_{m+1} < \delta \leqslant \beta_m$. Then, using (b),

$$\omega(p;f;\delta) \leqslant 4K^{1/2}CB'\delta \sum_{k=1}^{m} \beta_k^{\alpha-1} + 2B'\delta^\alpha \sum_{k=m+1}^{\infty} \left(\frac{\beta_k}{\beta_{m+1}}\right)^\alpha$$
$$\leqslant 4K^{1/2}CB'\delta \sum_{k=1}^{m} \beta_k^{\alpha-1} + \frac{2B'}{1-\mu^\alpha}\delta^\alpha.$$

The estimates in the statement of the theorem now follow easily as in the proof of [1, Theorem 1]. □

We shall determine some metric groups for which families $\{\Upsilon_n\}_{n=1}^\infty$, (β_n) can be found satisfying (a)–(c) above.

The classical examples are the real line **R** and the circle group **T**. For $G = \mathbf{R}$, just put $\Upsilon_n = [-\beta_n^{-1}, \beta_n^{-1}]$ for any sequence (β_n) satisfying (b) (here we identify $\Gamma_{\mathbf{R}}$ with **R**). The case $G = \mathbf{T}$ is analogous.

It was shown in [1] that every locally compact metric 0-dimensional abelian group G will admit such families; here we take (β_n) to be any sequence of positive numbers smaller than 1 that satisfy (b) for some $\mu \in (0,1)$, and put

$$\Upsilon_n = A(\Gamma, \{x \in G: d(x,0) < \beta_n\}).$$

Other examples of groups admitting families $\{\Upsilon_n\}_{n=1}^\infty$, (β_n) as above will be obtained from finite products and homomorphic images of 0-dimensional groups and the real line. In general for finite products we have

THEOREM 2. *Suppose* (G,d), (G',d') *are metric LCA groups that admit families* $\{\Upsilon_n\}_{n=1}^\infty$, (β_n), $\{\Upsilon'_n\}_{n=1}^\infty$, (β'_n) *respectively, each satisfying* (a)–(c) *of Theorem* 1. *Then* $G \times G'$ *also satisfies* (a)–(c) *with the families* $\{[\Upsilon_n, \Upsilon'_n]\}_{n=1}^\infty$, $(\min\{\beta_n, \beta'_n\})$, *and* $\mu'' = \max\{\mu, \mu'\}$.

PROOF. Property (a) follows easily from

$$|[\gamma,\gamma']((a,a')) - 1| = |\gamma(a)\gamma'(a') - 1| \leqslant |\gamma(a) - 1| + |\gamma'(a') - 1|$$
$$\leqslant C\beta_n^{-1} d(a,0) + C'\beta_n'^{-1}\, d(a',0')$$
$$\leqslant (C + C')\max\{\beta_n^{-1}, \beta_n'^{-1}\} d''((a,a'),(0,0'))$$

for $[\gamma,\gamma'] \in [\Upsilon_n, \Upsilon'_n]$. The proof of (b) is trivial. For (c) we make use of the uniqueness property of Haar measure to obtain

$$\theta''(3[\Upsilon_n, \Upsilon'_n]) = \theta''([3\Upsilon_n, 3\Upsilon'_n]) = \eta\theta(3\Upsilon_n)\theta'(3\Upsilon'_n)$$
$$\leqslant \eta KK'\theta(\Upsilon_n)\theta'(\Upsilon'_n) = KK'\theta''([\Upsilon_n, \Upsilon'_n]),$$

where η is some positive constant. □

Our corresponding result for homomorphic images is a little more restrictive; here we consider G/H, where H is a *compact* subgroup of G.

THEOREM 3. *Let* G *be a metric LCA group that admits families* $\{\Upsilon_n\}_{n=1}^\infty$, (β_n) *satisfying* (a)–(c) *of Theorem* 1. *If* H *is a compact subgroup of* G *then* G/H *also satisfies* (a)–(c) *with the families* $\{(\Upsilon_n \cap A(\Gamma, H))^+\}_{n=1}^\infty$, (β_n) *and the same constant* μ.

PROOF. To show that (a) is satisfied, consider $\gamma^+ \in (\Upsilon_n \cap A(\Gamma, H))^+$ and $a \in G$. For any $b \in a + H$,

$$|\gamma^+(a+H) - 1| = |\gamma^+(b+H) - 1| = |\gamma(b) - 1| \leqslant C\beta_n^{-1} d(b,0).$$

Hence

$$|\gamma^+(a+H) - 1| \leqslant C\beta_n^{-1} \inf\{d(b,0)\colon b \in a + H\} = C\beta_n^{-1} d^*(a+H, H).$$

Obviously (b) holds with the same (β_n), μ. For the proof of (c) we require H to be compact, so that $A(\Gamma, H)$ is open; here, for some positive constant η,

$$\begin{aligned}\theta^*(3(\Upsilon_n \cap A(\Gamma,H))^+) &= \theta^*((3(\Upsilon_n \cap A(\Gamma,H)))^+) = \eta\theta(3(\Upsilon_n \cap A(\Gamma,H))) \\ &\leqslant \eta K\theta(\Upsilon_n \cap A(\Gamma,H)) = K\theta^*((\Upsilon_n \cap A(\Gamma,H))^+),\end{aligned}$$

as required. □

Our main example of a finite dimensional compact abelian group, namely the $\mathbf{a}$-adic solenoid $\Sigma_{\mathbf{a}}$, is not covered by Theorems 2 and 3. We define $\Sigma_{\mathbf{a}}$ by

$$\Sigma_{\mathbf{a}} = (\mathbf{R} \times \Delta_{\mathbf{a}})/B,$$

where B is the cyclic discrete subgroup of $\mathbf{R} \times \Delta_{\mathbf{a}}$ generated by $(1, \mathbf{u})$, $\mathbf{u} = (1, 0, 0, \ldots)$.

A metric d^* will be given for $\Sigma_{\mathbf{a}}$ according to the specifications in the beginning of this paper. We assert that the families $\{\Upsilon_n\}_{n=1}^{\infty}$, (β_n) satisfy (a)–(c), where

$$\Upsilon_n = ([[-\beta_n^{-1}, \beta_n^{-1}], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B))^+.$$

Now Theorems 2, 3 apply to show that (a), (b) hold. To prove that (c) holds for the above choice of $\{\Upsilon_n\}_{n=1}^{\infty}$, set

$$\kappa = \theta^*(([(0,1], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B))^+).$$

For each $m \in \mathbf{Z}$,

$$\begin{aligned}&\theta^*(([(m, m+1], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B))^+) \\ &\qquad = \theta^*([m,0]^+ + ([(0,1], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B))^+) = \kappa,\end{aligned}$$

where we have used the translation-invariance of θ^* (note that $[m, 0] \in A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B)$). Hence

$$\begin{aligned}\frac{\theta^*(3\Upsilon_n)}{\theta^*(\Upsilon_n)} &\leqslant \frac{\theta^*(([[-3\beta_n^{-1}, 3\beta_n^{-1}], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B))^+)}{\theta^*(([[-\beta_n^{-1}, \beta_n^{-1}], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R}\times\Delta_{\mathbf{a}}}, B))^+)} \\ &\leqslant (6\beta_n^{-1} + 2)/(2\beta_n^{-1} - 2) < 4\end{aligned}$$

for n suitably large, which is all we need to prove. □

Now Theorem 4 of [3] can be modified to give the following analogue of Jackson's theorem for $\Sigma_{\mathbf{a}}$:

THEOREM 4. *For each* $n \in \{1, 2, \ldots\}$ *put*

$$\Omega_n = \left\{ \frac{l}{a_0 a_1 \cdots a_{n-1}} : l \in \mathbf{Z} \text{ and } \left| \frac{l}{a_0 a_1 \cdots a_{n-1}} \right| \leqslant \beta_n^{-1} \right\}$$

(*here we are identifying the character group of* $\Sigma_{\mathbf{a}}$ *with a subgroup of the group* $\mathbf{Q}$ *of rational numbers*). *Then there is a constant* K *such that*

$$E_{\Omega_n}(p; f) \leqslant K\omega(p; f; \pi_B((-\beta_n, \beta_n) \times \Lambda_n))$$

for every $f \in L^p(\Sigma_{\mathbf{a}})$ *if* $p \in [1, \infty)$, *or for every continuous* f *if* $p = \infty$.

It follows that if $f \in \operatorname{Lip}_p \alpha$ for some $\alpha > 0$ then

$$E_{\Omega_n}(p; f) = O(\beta_n^{\alpha}). \tag{4}$$

Combining (4) with Theorem 1, and observing that

$$\Omega_n = ([[-\beta_n^{-1}, \beta_n^{-1}], A(\Gamma_{\Delta_{\mathbf{a}}}, \Lambda_n)] \cap A(\Gamma_{\mathbf{R} \times \Delta_{\mathbf{a}}}, B))^{+}$$

then, under the further assumption that $\beta_{n+1} = \mu\beta_n$ for $n = 1, 2, \ldots,$ we have for the $\mathbf{a}$-adic solenoid:

THEOREM 5. *Let* $\alpha \in (0, 1)$ *be given. Then* $f \in \operatorname{Lip}_p \alpha$ *if and only if* $E_{\Omega_n}(p; f) = O(\beta_n^{\alpha})$ *where, for* $p = \infty$, f *is taken to be continuous.*

Obtaining an analogous result for finite dimensional groups is a little more involved. First we see from [5, Lemma 1] that a finite dimensional compact metric abelian group is topologically isomorphic with $(\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G})/H$, where $\mathbf{a}$ is chosen so that $\mathbf{Q}$ is the character group of $\Sigma_{\mathbf{a}}$, $\Delta_{\mathbf{a}}^{\infty}$ is the direct product of countably many copies of $\Delta_{\mathbf{a}}$, $\dim G$ is the (finite) topological dimension of G, and H is a closed 0-dimensional subgroup of $\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G}$. Note that $\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G}$ is compact, and hence so is H.

Now write

$$\mathcal{W}_n = \{\mathbf{x} \in \Delta_{\mathbf{a}}^{\infty} : x_{i1} = x_{i2} = \cdots = x_{in} = 0, i = 1, 2, \ldots, n\},$$

$$U_n = \mathcal{W}_n \times (\pi_B((-\beta_n, \beta_n) \times \Lambda_n))^{\dim G}, \text{ and}$$

$$V_n = \pi_H(U_n).$$

Let $\mathcal{U}_n$ (respectively $\mathcal{V}_n$) be the open subgroup of $\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G}$ (respectively $(\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G})/H$) generated by U_n (respectively V_n) (note that $\mathcal{U}_n = \mathcal{W}_n \times \Sigma_{\mathbf{a}}^{\dim G}$) and set

$$\nabla_n = \sigma_{\mathcal{V}_n}^{-1}[\tilde{\nu}_n((\sigma_{\mathcal{U}_n}([A(\Gamma_{\Delta_{\mathbf{a}}^{\infty}}, \mathcal{W}_n), [\Omega_n]^{\dim G}]) \cap A(\Gamma_{\mathcal{U}_n}, \mathcal{U}_n \cap H))^{+})];$$

here we use the notation that for an open subgroup $\mathcal{U}$ and a closed subgroup H of an LCA group G, $\sigma_{\mathcal{U}}$ denotes the restriction map of Γ_G onto $\Gamma_{\mathcal{U}}$, and $\tilde{\nu}$ denotes the adjoint of the natural topological isomorphism

$$\nu: \pi_H(\mathcal{U}) \to \mathcal{U}/(\mathcal{U} \cap H).$$

With these definitions we have, from [3, Theorems 1–3], the existence of a constant K such that

$$(5) \qquad E_{\nabla_n}(p;f) \leqslant K\omega(p;f;V_n)$$

for every $f \in L^p((\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G})/H)$ if $p \in [1,\infty)$, or for every continuous f if $p = \infty$.

To match this result with Theorem 1 we require that

$$(6) \qquad \nabla_n = ([A(\Gamma_{\Delta_{\mathbf{a}}^{\infty}}, \mathcal{W}_n), [\Omega_n]^{\dim G}] \cap A(\Gamma_{\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G}}, H))^{+}.$$

As part of the proof of (6) we appeal to the following general result:

LEMMA. *Let G be an LCA group, with a closed subgroup H and an open subgroup $\mathcal{U}$. Then, for any $\Upsilon \subset \Gamma_G$,*

$$(7) \qquad \sigma_{\pi_H(\mathcal{U})}((\Upsilon \cap A(\Gamma_G, H))^{+}) \subset \nu^{\sim}((\sigma_{\mathcal{U}}(\Upsilon) \cap A(\Gamma_{\mathcal{U}}, \mathcal{U} \cap H))^{+}).$$

PROOF. First notice that both sides of the above inclusion are subsets of $\Gamma_{\pi_H(\mathcal{U})}$; just use [4, (24.5)] and the property that $\pi_H(\mathcal{U})$ is an open subgroup of G/H.

Let $\chi \in \Upsilon \cap A(\Gamma_G, H)$ and $x \in \mathcal{U}$. Then

$$\sigma_{\pi_H(\mathcal{U})}(\chi^{+})(x + H) = \chi^{+}(x + H) = \chi(x).$$

Also $\sigma_{\mathcal{U}}(\chi) \in A(\Gamma_{\mathcal{U}}, \mathcal{U} \cap H)$ and (recall that $x \in \mathcal{U}$)

$$\begin{aligned} \nu^{\sim}(\sigma_{\mathcal{U}}(\chi)^{+})(x + H) &= \sigma_{\mathcal{U}}(\chi)^{+}(\nu(x + H)) = \sigma_{\mathcal{U}}(\chi)^{+}(x + \mathcal{U} \cap H) \\ &= \sigma_{\mathcal{U}}(\chi)(x) = \chi(x), \end{aligned}$$

so that $\sigma_{\pi_H(\mathcal{U})}(\chi^{+}) = \nu^{\sim}(\sigma_{\mathcal{U}}(\chi)^{+})$. □

Unfortunately the inclusion reverse to that in (7) does not seem to hold in general. However we can establish (the special case) (6) quite easily as follows; consider $\eta \in \nabla_n$ and write $\eta = \gamma^{+}$, where $\gamma \in A(\Gamma_{\Delta_{\mathbf{a}}^{\infty} \times \Sigma_{\mathbf{a}}^{\dim G}}, H)$. We know that $\gamma = [\gamma_0, \gamma_1, \ldots, \gamma_d]$ for some $\gamma_0 \in \Gamma_{\Delta_{\mathbf{a}}^{\infty}}$ and $\gamma_1, \ldots, \gamma_d \in \Gamma_{\Sigma_{\mathbf{a}}}$ ($d = \dim G$). Since $\sigma_{\pi_H(\mathcal{U}_n)}(\eta)$ can be identified with an element of

$$\sigma_{\mathcal{U}_n}([A(\Gamma_{\Delta_{\mathbf{a}}^{\infty}}, \mathcal{W}_n), [\Omega_n]^{\dim G}]) = [\{0\}, [\Omega_n]^{\dim G}],$$

it is apparent that $\gamma_0 \in A(\Gamma_{\Delta_{\mathbf{a}}^{\infty}}, \mathcal{W}_n)$ and $\gamma_1, \ldots, \gamma_d \in \Omega_n$, that is

$$\gamma \in [A(\Gamma_{\Delta_{\mathbf{a}}^{\infty}}, \mathcal{W}_n), [\Omega_n]^{\dim G}];$$

this gives the required result.

Now that we know that (6) holds we can appeal to (5) and Theorems 1, 2, and 3 (the metric on $\Delta_{\mathbf{a}}^{\infty}$ is chosen with respect to the basis $(\mathcal{W}_n)$) to obtain, once more under the assumption that $\beta_{n+1} = \mu\beta_n$ for $n = 1, 2, \ldots$:

THEOREM 6. *Let G be a finite dimensional compact metric abelian group. With the notation above we have that for $\alpha \in (0,1)$ given, $f \in \mathrm{Lip}_p\,\alpha$ if and only if $E_{\nabla_n}(p;f) = O(\beta_n^{\alpha})$ where, for $p = \infty$, f is taken to be continuous.*

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