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A PROOF OF THE COMPACT LEAF CONJECTURE FOR FOLIATED BUNDLES

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ABSTRACT. Given an oriented fiber bundle M whose fiber is a connected, m-dimensional manifold, and a codimension n foliation of M which is transverse to the fibers of M and all of whose leaves are compact, we will show that there is an upper bound on the orders of the holonomy groups of the leaves.

In [1], the authors assert the following: If M is a compact, m-dimensional manifold, and $\mathfrak{F}$ is a codimension n foliation of M, all of whose leaves are compact, then there is an integer $r > 0$ such that the order of $H(L)$, the holonomy group of the leaf L, is less than or equal to r, for all leaves L of the foliation. The proof, however, was incomplete; in fact, Sullivan [6] has recently constructed a counterexample to the above assertion, in codimension four. The Compact Leaf Conjecture, at the present, is unknown in the codimension three case, but has been shown to be true in codimensions one and two [2]. Sullivan's codimension four counterexample can be used to construct counterexamples in all higher codimensions.

On the other hand, if M is assumed to be a fiber bundle, where the fiber N is connected and of dimension n, but not necessarily compact, and if the foliation is transverse to the fibers of M and all of its leaves are compact, then the conjecture is true. The proof uses the following two results, due to Montgomery and Yang [4].

LEMMA. *Let G be a discrete group acting on a connected manifold N, such that every orbit is finite, and let $F(G) = \{x \in N: G(x) = x\}$. If $F(G)$ contains an interior point, then $F(G) = N$.*

PROOF. Each element of G is a pointwise periodic, hence periodic, transformation of N. But a periodic transformation which leaves fixed an open set in N must fix all of N. Both of these statements are proved in [5, pp. 223–224].

THEOREM 1. *Let G be a discrete group acting on a connected manifold N, such that every orbit is finite. If G is finitely generated and acts effectively, then G is finite.*

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PROOF. It is known that G, as above, contains only a finite number of subgroups of finite index j [3]. Since every orbit of the action is finite, and since $|Gs| = |G \backslash G_s|$, it follows that all the isotropy subgroups have finite index in G; here, Gs and G_s denote the orbit of s and the isotropy subgroup of s, respectively, and $|\ |$ denotes cardinality. Thus, the isotropy subgroups form a countable collection which we will denote by H_{ij}. The set $F(H_{ij})$ is closed, and $N = \bigcup F(H_{ij})$; thus, by the Baire Category Theorem, it follows that for some $i_1 j_1$, $F(H_{i_1 j_1})$ must contain an interior point. Hence, the lemma implies that $F(H_{i_1 j_1}) = N$. Since G acts effectively, $H_{i_1 j_1}$ is the identity subgroup. As it is also a subgroup of finite index, G must be finite.

Now let M be an oriented fiber bundle with base B and n-dimensional fiber N, which is connected, and let $\mathfrak{F}$ be a codimension n foliation of M which is transverse to the fibers and all of whose leaves are compact. Then $\mathfrak{F}$ is defined by an action of $G = \pi_1(B)$ on N; G is finitely generated and, by hypothesis, all of the G-orbits are finite. Let $H = \{g \in G\colon g(x) = x, \text{ for all } x \in N\}$; then H is a normal subgroup of G, and thus the action of G on N naturally defines an action of G/H on N. This latter action is effective, so Theorem 1 implies that G/H is finite. But the G-action and the G/H-action define the same foliated bundles; hence, even though G may be infinite, the cardinalities of the G-orbits are bounded above (by the order of G/H). In terms of the holonomy of the leaves, this says that the order of $H(L)$ is bounded above, for all leaves L of $\mathfrak{F}$. This concludes the proof of the Compact Leaf Conjecture for foliated bundles.

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