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PROJECTIVE MODULES

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ABSTRACT. In this note we prove that if R is a ring satisfying a polynomial identity and P is a projective left R-module such that P is finitely generated modulo the Jacobson radical, then P is finitely generated. As a corollary we get that if R is a ring still satisfying a polynomial identity and M is a finitely generated flat R-module such that M/JM is R/J-projective, then M is R-projective, J denotes the Jacobson radical.

0. **Introduction.** In this note R denotes an associative ring with an identity element, J denotes the Jacobson radical of R and all modules considered are unitary R-modules.

In [6] D. Lazard proved that a projective module P over a commutative ring R, such that P/JP is finitely generated, is finitely generated. Furthermore he remarked that is was unknown whether or not the commutativity of R was essential for the validity of the result. In this note we prove that the theorem holds in any P.I. ring.

1. **Flat and projective modules.** Let us consider the following properties of a ring R.

(i) Any projective left R-module P, such that P/JP is finitely generated, is finitely generated.

(ii) Any finitely generated submodule of a projective module is contained in a maximal submodule.

(iii) Any finitely generated flat left R-module M, such that M/JM is R/J-projective, is projective.

First we prove that if (i) holds for all projective left R-modules, then (iii) will also hold. This result might be well known, but I have not been able to find it in the literature.

PROPOSITION 1.1. *If* (i) *holds in the ring* R, *then* (iii) *will also hold.*

PROOF. Suppose M is a finitely generated flat left R-module, such that M/JM is R/J-projective, and M is not projective. We have an exact sequence $0 \to K \to F \to M \to 0$ where F is free of finite rank and K is not finitely generated. (This follows since a finitely generated flat module is projective if and only if it is finitely related.)

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It follows from the assumptions that K/JK is finitely generated. Consequently there exists a finitely generated submodule of K, K_0, say, such that $K = K_0 + JK$. It follows readily from D. Lazard [5, Chapitre I, Théorème 3.1] that there exists a countably, but not finitely generated, submodule K_1 of K, such that $K_0 \subseteq K_1$ and $K_1 = K_0 + JK_1$, with F/K_1 R-flat. K_1 is projective by the proof of Jensen's lemma [4, Lemma 2] or [5, Chapitre I, Théorème 3.2]. By (i), K_1 is finitely generated, thus we have obtained a contradiction.

(i) is known to hold in any commutative ring [6, Proposition 5]. (iii) was proved for commutative rings in [8, Theorem 2.1].

PROPOSITION 1.2. *Condition* (i) *is equivalent to* (ii).

PROOF. (ii) implies (i). Suppose A is a finitely generated submodule of a projective module P. By (ii), A is contained in a maximal submodule B. JP is the intersection of the maximal submodules of P [3, p. 180, Exercise 8], hence $JP \subseteq B$, so $P \neq A + JP$.

(i) implies (ii). If $(f_i, p_i)_{i \in I}$ is a set of projective coordinates for P, then for some finite set I_0, $f_i(A) = 0$, $i \notin I_0$. Letting C be the submodule generated by the p_i's, $i \in I_0$, (i) implies $P \neq C + JP$ (unless P is finitely generated, in which case (ii) is true), so $f_i(P) \nsubseteq J$ for some $i \notin I_0$. It follows that $f_i(P)$ has a maximal submodule, whose inverse image is a maximal submodule of P containing A.

Proposition 1.2 as well as the proof is due to the referee.

2. **Projective modules.** Before we state and prove the next result let us recall a theorem due to I. Beck [2, Theorem 6]:

Let P be a projective left R-module. If P/JP is a free R/J-module, then P is a free R-module.

PROPOSITION 2.1. *Let I be a two-sided ideal contained in the prime radical of R. If P is a projective left R-module such that P/IP is finitely generated, then P is finitely generated.*

PROOF. By using the Morita equivalence between R and the matrix ring $M_n(R)$ it follows that we can assume without loss of generality that P/IP is cyclic (all assumptions will be carried over to $M_n(R)$ under the equivalence). Since P/IP is cyclic and R/I-projective, P/IP is isomorphic to $(R/I)(\bar{e})$, where $\bar{e}$ is an idempotent element in R/I. We lift $\bar{e}$ to an idempotent e in R; this is possible since I is a nil ideal. The module $P \oplus R(1 - e)$ must be free by Beck's theorem mentioned above, thus $P \oplus R(1 - e)$ is finitely generated so P must also be finitely generated.

The following lemma (and the corollary) was kindly communicated to me by G. M. Bergman.

LEMMA 2.2. *Let R be a ring and f: $P \to P'$ a homomorphism of projective left R-modules. If the induced map $\bar{f}$: $P/JP \to P'/JP'$ is left invertible, then f is a monomorphism.*

PROOF. Let $P \oplus Q$ be free, then replacing f by $f \oplus 1_Q \colon P \oplus Q \to P' \oplus Q$, we are reduced to the case where P is free. Since P' is a direct summand in a free module, we may further take P' to be free, i.e. f to be a map of free modules. Now, since P is the union of its finitely generated summands, we may replace P by one of these summands and so assume it to be finitely generated. If we replace P' similarly by a finitely generated direct summand containing $f(P)$, we see that we have reduced to the case where P' is also finitely generated, and now it is well known that $\bar{f}$ left invertible implies f left invertible.

REMARK. We cannot conclude by this argument that the original map f was left invertible–one of the reductions that we made does not reflect left-invertibility, namely, going to a class of direct summands whose union is P.

Now note that if P/JP is finitely generated (P is still projective), then P/JP can be written as a direct summand of a free left R/J-module of finite rank. Applying to the injection $\bar{f}\colon P/JP \to (R/J)^n$, we get

COROLLARY 2.3. *Let R be a ring, and P a projective left R-module. If P/JP may be generated by n elements, then P can be embedded in the free left R-module R^n.*

(The case $n = 0$ is a result of H. Bass [1, Proposition 2.7].)

LEMMA 2.4. *Let P be a projective left R-module, and assume P/JP is finitely generated. If for every prime ideal Q in R, P/QP is finitely generated, then P is finitely generated.*

PROOF. By Proposition 2.1 it suffices to prove that P/NP is finitely generated, where N denotes the intersection of all prime ideals (the prime radical).

We will prove a slightly more general result, namely the following: Let P be a projective module, if P/JP is finitely generated and for some family of two-sided ideals I_γ, $\gamma \in \Gamma$, $P/I_\gamma P$ is finitely generated, then $P/(\bigcap I_\gamma)P$ is finitely generated. It is easily seen that $P/(I_\gamma \cap J)P$ is finitely generated, hence it follows that we can assume that all the I_γ's are contained in J.

Since P/JP is finitely generated there exists a finitely generated submodule P_0 of P, such that $P = P_0 + JP$. If P/IP is finitely generated for some two-sided ideal $I \subseteq J$, then by the Nakayama lemma $P = P_0 + IP$. Now let $(p'_1, \ldots, p'_k)$ be a set of generators for P_0 and let us choose a set of projective coordinates $(f_s, p_s)_{s \in S}$ for the projective module P, such that $p'_t \in \{p_s | s \in S\}$. For any $p \in P$ we let supp (p) denote the finite set $\{s \in S | f_s(p) \neq 0\}$.

For any $p \in P$ and for any $\gamma \in \Gamma$, we get

$$f_s(p) = f_s\Big(\sum_t r_t p'_t + x\Big), \qquad x \in I_\gamma P.$$

Thus for all $p \in P$, $f_s(p) \in I_\gamma$ for all $s \notin \bigcup_t \operatorname{supp}(p'_t) = S_0$, say. We note that S_0 is finite. We get for any $p \in P$

$$p = \sum_{s \in S_0} f_s(p)p_s + \sum_{s \notin S_0} f_s(p)p_s \in P_1 + \left(\bigcap_{\gamma} I_\gamma\right)P$$

where P_1 denotes the finitely generated module generated by the p_s's, $s \in S_0$. Hence it follows that $P/(\bigcap_\gamma I_\gamma)P$ is finitely generated and the proof of the proposition is completed.

Note that if P/I_1P is finitely generated and $I_1 \subseteq I_2$, then P/I_2P is also finitely generated. Thus to prove that (i) holds in a ring R it suffices to prove that (i) holds in any prime factor ring.

THEOREM 2.5. *Let R be a P.I. ring and P a projective left R-module such that P/JP is finitely generated, then P is finitely generated.*

PROOF. By Corollary 2.3 we may take P to be a submodule of R^n for some $n \in N$. Under the Morita equivalence between R and the matrix ring $M_n(R)$, the module P corresponds to a module isomorphic to a left ideal. $M_n(R)$ is again a P.I. ring, and by Lemma 2.4 and the remark below it follows that we can assume that R is a prime P.I. ring and P a projective ideal of R. The proof of the theorem is now completed by the following lemma.

LEMMA 2.6. *Any projective left (or right) ideal in a prime P.I. ring is finitely generated.*

PROOF. Lemma 2.6 follows easily from [7, Theorem 2.1], since a prime P.I. ring is a Goldie ring.

The author wishes to thank the referee for pointing out that Lemma 2.6 is an easy consequence of Sandomierski's result and for the following remark:

If R is a ring such that all prime factors are Goldie, then (i) will hold in R.

Hence the proof of Theorem 2.5 applies to a larger class of rings then P.I. rings.

COROLLARY. *Let R be a ring with all prime factors Goldie and M a finitely generated flat R-module, such that M/JM is R/J-projective. Then M is R-projective.*

FURTHER REMARKS. It follows from Corollary 2.3 that any left noetherian ring will satisfy (i).

If R is a left semihereditary ring it is known that every projective left (or right) R-module is a direct sum of finitely generated left ideals. Thus, by the Nakayama lemma, any left semihereditary ring will satisfy (i).

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