EXISTENCE OF A FIXED POINT FOR NONEXPANSIVE MAPPINGS WITH CLOSED VALUES

J. T. MARKIN

Abstract. Fixed point existence and fixed point stability results are presented for nonexpansive mappings of a Banach space $B$ into the family of nonempty closed bounded convex subsets of $B$, where $B$ is assumed separable, strictly convex, and reflexive with a weakly continuous duality mapping.

The study of the existence of a fixed point for nonexpansive set valued mappings was initiated in [7] for Hilbert space and extended in [6] to Banach spaces satisfying Opial's condition and in [5] to strictly convex reflexive Banach spaces with weakly continuous duality mapping. All of these results have assumed the mappings have compact values. In Theorem 1 the nonexpansive mapping is assumed to have closed bounded convex values and existence of a fixed point is shown for separable strictly convex reflexive Banach spaces with weakly continuous duality mapping. This class of spaces includes separable Hilbert spaces and the $l^p$ spaces, $1 < p < \infty$.

The family of nonempty closed bounded convex subsets of a Banach space $B$ is denoted by $K(B)$. Let $D$ denote the Hausdorff metric defined on the closed bounded subsets of $B$, which is generated by the norm $\|\cdot\|$ of $B$. A mapping $F$ of $B$ into $K(B)$ is nonexpansive if $D(F(x), F(y)) < \|x - y\|$ for $x, y \in B$.

A mapping $J$ of a Banach space $B$ into its dual $B^*$ is a duality mapping if $(x, J(x)) = \|x\| \|J(x)\|$ and $\|J(x)\| = \mu(\|x\|)$ for $x \in B$, where $\mu$ is a nonnegative nondecreasing function on $R^1$ with $\mu(0) = 0$. A duality mapping $J$ is said to be weakly continuous if it is continuous from $B$ with the weak topology into $B^*$ with the weak*-topology. Weak convergence of a sequence $\{x_n\}$ to a point $x$ is denoted by $x_n \rightharpoonup x$.

A mapping $F$ of a Banach space $B$ into itself is $J$-monotone provided for any pair $x, y \in B$ and $x_1 \in F(x)$ there is a $y_1 \in F(y)$ such that $(x_1 - y_1, J(x - y)) \geq 0$, where $J$ is a duality mapping on $B$.

For any mapping $F$ of $B$ into the nonempty subsets of $B$ and any subset $C$ of $B$, $F(C)$ denotes $\bigcup_{x \in C} F(x)$. A point $y \in B$ is a fixed point of $F$ if $y \in F(y)$.
The following two lemmas extend similar results in [5] for compact valued mappings.

**Lemma 1.** Let $B$ be a strictly convex reflexive Banach space with a weakly continuous duality mapping $J$, and $F$ a continuous mapping of $B$ with the norm topology into $\mathcal{K}(B)$ with the Hausdorff metric $D$. If for a given pair $x, x_1 \in B$ and any $y \in B$ there is a $y_1 \in F(y)$ such that $(x_1 - y_1, J(x - y)) \geq 0$, then $x_1 \in F(x)$.

**Proof.** Let $x, x_1$ be elements of $B$ such that for any $y \in B$ there is $y_1 \in F(y)$ satisfying $(y_1 - x_1, J(y - x)) \geq 0$. Suppose $x_1 \notin F(x)$. Since $F(x)$ is weakly compact and convex there is a continuous linear functional $w$ strictly separating $x_1$ and $F(x)$; i.e., $(x_1, w) < (z, w)$ for $z \in F(x)$. Ko [5] has shown that if $B$ is reflexive then a weakly continuous duality mapping maps $B$ onto $B^*$ and therefore $w = J(u)$ for some $u \in B$. Hence,

\[(1) \quad (x_1, J(u)) < (z, J(u))\]

for $z \in F(x)$.

Setting $u_n = x - u/n$, $n = 1, 2, \ldots$, there is by assumption for each $u_n$ a $z_n \in F(u_n)$ such that

\[(2) \quad (x_1 - z_n, J(x - u_n)) = (x_1 - z_n, J(u/n)) \geq 0.\]

By a result of Browder [3] the strict convexity of the norm of $B$ implies that $J(u/n) = J(u)/n$. Inequality (2) can then be written as

\[(3) \quad (x_1 - z_n, J(u)) \geq 0\]

for each $n$.

By the continuity of $F$, $D(F(u_n), F(x))$ tends to 0 and therefore we may assume that $(z_n)$ converges weakly to a point $z_0$ and that there is a sequence $(y_n)$, $y_n \in F(x)$ for which $\lim_{n \to \infty} \|z_n - y_n\| = 0$, where $y_n \rightharpoonup y_0 \in F(x)$. We assert that $z_0 \in F(x)$ so that by inequality (3), $(x_1, J(u)) \geq (z_0, J(u))$, contradicting (1). Indeed, if $z_0 \notin F(x)$ there is a continuous linear functional $v$ such that $(z_0, v) < (y_0, v)$ and hence

\[0 > (z_0 - y_0, v) = (z_0 - z_n, v) + (z_n - y_n, v) + (y_n - y_0, v).\]

The right side of the latter equality tends to 0, which is not possible.

**Lemma 2.** Let $B$ be a separable strictly convex reflexive Banach space with weakly continuous duality mapping $J$, $C$ a weakly compact subset of $B$, and $F$ a continuous $J$-monotone mapping of $B$ into $\mathcal{K}(B)$ with the Hausdorff metric $D$. Then $F(C)$ is closed.

**Proof.** Let $v_0$ lie in the closure of $F(C)$. Then there is a sequence $(v_i)$ such that $\lim_{i \to \infty} v_i = v_0$, where $v_i \in F(u_i)$, $u_i \in C$ and, by weak compactness of $C$, it is assumed that $u_i \rightharpoonup u_0 \in C$. The assumption that $v_0 \notin F(C)$ will be shown to lead to a contradiction.
For some $x \in B$ it must be the case that there is a $\delta > 0$ such that

\[(z - v_0, J(x - u_0)) < -\delta\]

for each $z \in F(x)$; for otherwise Lemma 1 would imply that $v_0 \in F(u_0)$. For each nonnegative integer $j$ let $B_j = F(x) - v_j$. Since $v_j \to v_0$, the sequence $\{B_j\}$ converges to $B_0$ in the Hausdorff metric $D$.

Choose a closed ball $S$ in $B$ which contains the sets $\{B_j\}, j = 0, 1, \ldots$. By the reflexivity of $B$ the ball $S$ is weakly compact and by [4] the weak topology on $S$ is metrizable. Metrizing the weak topology on $S$ by

\[d(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n(a - b)|}{1 + |x_n(a - b)|}\]

where $\{x_n\}$ is a countable dense subset of the unit ball of $B^*$ and $a, b \in S$, it is easily seen that $\{B_j\}$ converges to $B_0$ in the Hausdorff metric $H$ generated by $d$. Indeed,

\[d(a, b) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|a - b\|}{1 + \|a - b\|} \leq \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) \|a - b\|\]

implying that

\[H(A, B) \leq \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) D(A, B),\]

where $A, B$ are weakly closed subsets of $S$.

Define the functionals $\{L_i\}$ on $B$ by $L_i(y) = \sup_{x \in B} \langle y, J(x - u_i) \rangle, y \in B$. Since the functionals $\{(\cdot, J(x - u_j))\}$ are linear and, hence, convex, each $L_i$ is a convex functional [2].

Let the function $L$ denote an arbitrary member of $\{L_i\}$. We claim that $L$ is continuous on $S$ with the weak relative topology. Since $L$ is bounded on any norm bounded subset of $B$, assume that $L(y) \leq K$ for any $y \in S$. Let $N(0, \delta)$ be an origin centered open ball of radius $\delta$ in the weak metric topology on $S$. For $\alpha \in [0, 1)$ it is easily seen that $\alpha N(0, \delta) \subseteq N(0, \delta)$. If $y \in \alpha N(0, \delta)$ then there is a $z \in N(0, \delta)$ for which $y = \alpha z$ and therefore $(y/\alpha) \in N(0, \delta)$. By the convexity of $L$,

\[L((1 - \alpha)0 + \alpha(y/\alpha)) \leq (1 - \alpha)L(0) + \alpha L(y/\alpha).\]

Since $L(0) = 0$, $L(y) \leq \alpha L(y/\alpha) \leq \alpha K$. Hence, in the limit as $y$ approaches 0 in the weak topology $L(y) \leq L(0)$, which proves that $L$ is upper semicontinuous at 0 in the weak topology on $S$. Continuity of $L$ at 0 follows from the lower semicontinuity of the supremum of continuous functions and the definition of $L$. For any other point $y \in S$ appropriate translations reduce the problem to the case just considered where $y = 0$ and $L(y) = 0$.

Since $F$ is $J$-monotone, for each $u_j$ there is a $z_j \in F(x)$ such that $(z_j - v_j, J(x - u_j)) \geq 0$. Thus, for each $j > i$, $\sup_{y \in B} L_i(y) \geq 0$. The maximum theorem in [1] implies that
For each positive integer \( i \) let \( A_i = \{ y \in B_0 : L_i(y) \geq -1/i \} \). If \( y \in A_{i+1} \) then, by the definition of the \( L_i, L_i(y) \geq L_{i+1}(y) \geq -1/(i+1) > -1/i \), and therefore, \( y \in A_i \). Thus, \( A_{i+1} \subseteq A_i \) and, since the \( A_i \) are weakly closed subsets of the weakly compact set \( B_0 \), there is a point \( y_0 \in \cap_{i=1}^{\infty} A_i \). By the definition of \( B_0, y_0 = z_0 - v_0 \) where \( z_0 \in F(x) \), and by (5) for some subsequence \( \{u_k\} \) of \( \{u_i\} \), \((z_0 - v_0, J(x - u_k)) \geq -1/k \). Taking the limit in the latter inequality we have \((z_0 - v_0, J(x - u_0)) \geq 0\), which contradicts (4).

**Theorem 1.** Let \( B \) be a separable strictly convex reflexive Banach space with a weakly continuous duality mapping \( J \), and \( C \) a closed bounded convex subset of \( B \). If \( G \) is a nonexpansive mapping of \( B \) into \( K(B) \) with the Hausdorff metric \( D \), which maps \( C \) into itself, then \( G \) has a fixed point in \( C \).

**Proof.** Assume without loss of generality that \( 0 \in C \) [5]. The proof consists of showing that \( 0 \) is in the closure of \((I - G)(C)\) and that \( I - G \) is \( J \)-monotone. The theorem then follows by Lemma 2.

Let \( \{k_i\} \subseteq [0,1) \) be a sequence which converges to 1, and consider the sequence of mappings \( \{k_iG\} \) of \( C \) into \( K(C) \). By a result of Nadler [8] each \( k_iG \) has a fixed point \( x_{k_i} \in C \). Since \( x_{k_i} \in k_iG(x_{k_i}) \) we have \( x_{k_i} = k_iy_{k_i} \), where \( y_{k_i} \in G(x_{k_i}) \). Therefore,

\[
\inf_{y \in G(x_{k_i})} \|y - x_{k_i}\| \leq \|x_{k_i} - y_{k_i}\| \leq (1 - k_i)\|y_{k_i}\|
\]

and the last term tends to 0. This shows that \( 0 \) is in the closure of \((I - G)(C)\).

\( G \) being nonexpansive with closed convex values, given any \( y \in B \) and \( y_1 \in G(y) \), there is a closest point \( x_1 \in G(x) \) to \( y_1 \) such that \( \|x_1 - y_1\| \leq \|x - y\| \). It follows that

\[(x - x_1) - (y - y_1), J(x - y) \geq (\|x - y\| - \|x_1 - y_1\|)\|J(x - y)\| \geq 0,
\]

and hence \( I - G \) is \( J \)-monotone. Applying Lemma 2 we have \( 0 \in (I - G) \cdot (C) \); i.e., there is an \( x \in C \) such that \( x \in G(x) \).

**Theorem 2.** Let \( B \) be a separable strictly convex reflexive Banach space with a weakly continuous duality mapping \( J \), and \( C \) a closed bounded convex subset of \( B \). Assume that \( \{G_i\} \) is a sequence of nonexpansive mappings of \( B \) into \( K(B) \) with the Hausdorff metric, which converges pointwise to a nonexpansive mapping \( G_0 \) and maps \( C \) into itself. If \( x_i \in C \) is a fixed point of \( G_i, i = 1, 2, \ldots, \) and \( x_i \rightarrow x_0 \) then \( x_0 \) is a fixed point of \( G_0 \).

**Proof.** As in the proof of Theorem 1 the mappings \( I - G_i, i = 0, 1, \ldots, \) are \( J \)-monotone. Since \( x_i \) is a fixed point of \( G_i, 0 \in (I - G_i)(x_i) \) for \( i = 1, 2, \ldots, \), and by \( J \)-monotonicity for each \( v \in B \) there is a \( v_i \in (I - G_i)(v) \) for which

\[(u_i, J(v - x_i)) \geq 0, \quad i = 1, 2, \ldots.\]
Define the sequence of functionals \( \{L_i\} \) on \( B \) by

\[
L_i(y) = \sup_{j \geq i} (y, J(v - x_j)).
\]

As in the proof of Lemma 2 the \( \{L_i\} \) are continuous on \( S \) with the weak topology. Defining the sequence of weakly compact convex subsets \( \{B_j\} \) of \( B \) by \( B_j = (I - G_j)(v) \), the pointwise convergence of the \( \{G_j\} \) implies that \( \{B_j\} \) converges to \( B_0 = (I - G_0)(v) \) in the Hausdorff metric \( D \), and therefore, as shown in the proof of Lemma 2, the \( \{B_j\} \) converge to \( B_0 \) in the Hausdorff metric \( H \) generated by the weak metric topology on any ball containing the \( \{B_j\} \). Inequality (6) implies that for each \( i \), \( \sup_{y \in B_j} L_i(y) \geq 0 \) for \( j \geq i \).

Thus, the sequences \( \{L_i\} \) and \( \{B_j\} \) satisfy the same conditions as in the proof of Lemma 2, and therefore there is a point \( y_0 \in B_0 \) such that \( (y_0 - 0, J(v - x_0)) \geq 0 \). The point \( v \in B \) was arbitrary and so by Lemma 1, \( 0 \in (I - G_0)(x_0) \) and \( x_0 \) is a fixed point of \( G_0 \).

REFERENCES

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