EXISTENCE OF A FIXED POINT FOR NONEXPANSIVE MAPPINGS WITH CLOSED VALUES

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Abstract. Fixed point existence and fixed point stability results are presented for nonexpansive mappings of a Banach space $B$ into the family of nonempty closed bounded convex subsets of $B$, where $B$ is assumed separable, strictly convex, and reflexive with a weakly continuous duality mapping.

The study of the existence of a fixed point for nonexpansive set valued mappings was initiated in [7] for Hilbert space and extended in [6] to Banach spaces satisfying Opial's condition and in [5] to strictly convex reflexive Banach spaces with weakly continuous duality mapping. All of these results have assumed the mappings have compact values. In Theorem 1 the nonexpansive mapping is assumed to have closed bounded convex values and existence of a fixed point is shown for separable strictly convex reflexive Banach spaces with weakly continuous duality mapping. This class of spaces includes separable Hilbert spaces and the $l_p$ spaces, $1 < p < \infty$.

The family of nonempty closed bounded convex subsets of a Banach space $B$ is denoted by $K(B)$. Let $D$ denote the Hausdorff metric defined on the closed bounded subsets of $B$, which is generated by the norm $\|\cdot\|$ of $B$. A mapping $F$ of $B$ into $K(B)$ is nonexpansive if $D(F(x), F(y)) \leq \|x - y\|$ for $x, y \in B$.

A mapping $J$ of a Banach space $B$ into its dual $B^*$ is a duality mapping if $(x, J(x)) = \|x\| \|J(x)\|$ and $\|J(x)\| = \mu(\|x\|)$ for $x \in B$, where $\mu$ is a nonnegative nondecreasing function on $\mathbb{R}^1$ with $\mu(0) = 0$. A duality mapping $J$ is said to be weakly continuous if it is continuous from $B$ with the weak topology into $B^*$ with the weak*-topology. Weak convergence of a sequence $\{x_i\}$ to a point $x$ is denoted by $x_i \rightharpoonup x$.

A mapping $F$ of a Banach space $B$ into itself is $J$-monotone provided for any pair $x, y \in B$ and $x_1 \in F(x)$ there is a $y_1 \in F(y)$ such that $(x_1 - y_1, J(x - y)) \geq 0$, where $J$ is a duality mapping on $B$.

For any mapping $F$ of $B$ into the nonempty subsets of $B$ and any subset $C$ of $B$, $F(C)$ denotes $\bigcup_{x \in C} F(x)$. A point $y \in B$ is a fixed point of $F$ if $y \in F(y)$.

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The following two lemmas extend similar results in [5] for compact valued mappings.

**Lemma 1.** Let $B$ be a strictly convex reflexive Banach space with a weakly continuous duality mapping $J$, and $F$ a continuous mapping of $B$ with the norm topology into $K(B)$ with the Hausdorff metric $D$. If for a given pair $x, x_1 \in B$ and any $y \in B$ there is a $y_1 \in F(y)$ such that $(x_1 - y_1, J(x - y)) > 0$, then $x_1 \in F(x)$.

**Proof.** Let $x, x_1$ be elements of $B$ such that for any $y \in B$ there is $y_1 \in F(y)$ satisfying $(y_1 - x_1, J(y - x)) > 0$. Suppose $x_1 \notin F(x)$. Since $F(x)$ is weakly compact and convex there is a continuous linear functional $w$ strictly separating $x_1$ and $F(x)$; i.e., $(x_1, w) < (z, w)$ for $z \in F(x)$. Ko [5] has shown that if $B$ is reflexive then a weakly continuous duality mapping maps $B$ onto $B^*$ and therefore $w = J(u)$ for some $u \in B$. Hence,

$$\text{(1)} \quad (x_1, J(u)) < (z, J(u))$$

for $z \in F(x)$.

Setting $u_n = x - u/n, n = 1, 2, \ldots$, there is by assumption for each $u_n$ a $z_n \in F(u_n)$ such that

$$\text{(2)} \quad (x_1 - z_n, J(x - u_n)) = (x_1 - z_n, J(u/n)) \geq 0.$$

By a result of Browder [3] the strict convexity of the norm of $B$ implies that $J(u/n) = J(u)/n$. Inequality (2) can then be written as

$$\text{(3)} \quad (x_1 - z_n, J(u)) \geq 0$$

for each $n$.

By the continuity of $F$, $D(F(u_n), F(x))$ tends to 0 and therefore we may assume that $\{z_n\}$ converges weakly to a point $z_0$ and that there is a sequence $\{y_n\}, y_n \in F(x)$ for which $\lim_{n \to \infty} \|z_n - y_n\| = 0$, where $y_n \to y_0 \in F(x)$. We assert that $z_0 \in F(x)$ so that by inequality (3), $(x_1, J(u)) \geq (z_0, J(u))$, contradicting (1). Indeed, if $z_0 \notin F(x)$ there is a continuous linear functional $v$ such that $(z_0, v) < (y_0, v)$ and hence

$$\begin{align*}
0 &> (z_0 - y_0, v) = (z_0 - z_n, v) + (z_n - y_n, v) + (y_n - y_0, v).
\end{align*}$$

The right side of the latter equality tends to 0, which is not possible.

**Lemma 2.** Let $B$ be a separable strictly convex reflexive Banach space with weakly continuous duality mapping $J$, $C$ a weakly compact subset of $B$, and $F$ a continuous $J$-monotone mapping of $B$ into $K(B)$ with the Hausdorff metric $D$. Then $F(C)$ is closed.

**Proof.** Let $v_0$ lie in the closure of $F(C)$. Then there is a sequence $\{v_i\}$ such that $\lim_{i \to \infty} v_i = v_0$, where $v_i \in F(u_i), u_i \in C$ and, by weak compactness of $C$, it is assumed that $u_i \to u_0 \in C$. The assumption that $v_0 \notin F(C)$ will be shown to lead to a contradiction.
For some \( x \in B \) it must be the case that there is a \( \delta > 0 \) such that

\[
(z - v_0, J(x - u_0)) < -\delta
\]

for each \( z \in F(x) \); for otherwise Lemma 1 would imply that \( v_0 \notin F(u_0) \). For each nonnegative integer \( j \) let \( B_j = F(x) - v_j \). Since \( v_j \to v_0 \), the sequence \( \{B_j\} \) converges to \( B_0 \) in the Hausdorff metric \( D \).

Choose a closed ball \( S \) in \( B \) which contains the sets \( \{B_j\}, j = 0, 1, \ldots \). By the reflexivity of \( B \) the ball \( S \) is weakly compact and by [4] the weak topology on \( S \) is metrizable. Metrizing the weak topology on \( S \) by

\[
d(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n(a - b)|}{1 + |x_n(a - b)|}
\]

where \( \{x_n\} \) is a countable dense subset of the unit ball of \( B^* \) and \( a, b \in S \), it is easily seen that \( \{B_j\} \) converges to \( B_0 \) in the Hausdorff metric \( H \) generated by \( d \). Indeed,

\[
d(a, b) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|a - b\|}{1 + \|a - b\|} \leq \left( \sum_{n=1}^{\infty} \frac{1}{2^n} \right) \|a - b\|
\]

implying that

\[
H(A, B) \leq \left( \sum_{n=1}^{\infty} \frac{1}{2^n} \right) D(A, B),
\]

where \( A, B \) are weakly closed subsets of \( S \).

Define the functionals \( \{L_j\} \) on \( B \) by \( L_j(y) = \sup_{x \in B_j} (y, J(x - u_j)) \), \( y \in B \). Since the functionals \( \{(\cdot, J(x - u_j))\} \) are linear and, hence, convex, each \( L_j \) is a convex functional [2].

Let the function \( L \) denote an arbitrary member of \( \{L_j\} \). We claim that \( L \) is continuous on \( S \) with the weak relative topology. Since \( L \) is bounded on any norm bounded subset of \( B \), assume that \( L(y) \leq K \) for any \( y \in S \). Let \( N(0, \delta) \) be an origin centered open ball of radius \( \delta \) in the weak metric topology on \( S \). For \( \alpha \in [0, 1) \) it is easily seen that \( \alpha N(0, \delta) \subseteq N(0, \delta) \). If \( y \in \alpha N(0, \delta) \) then there is a \( z \in N(0, \delta) \) for which \( y = \alpha z \) and therefore \( (y/\alpha) \in N(0, \delta) \). By the convexity of \( L \),

\[
L((1 - \alpha)0 + \alpha(y/\alpha)) \leq (1 - \alpha)L(0) + \alpha L(y/\alpha).
\]

Since \( L(0) = 0 \), \( L(y) \leq \alpha L(y/\alpha) \leq \alpha K \). Hence, in the limit as \( \gamma \) approaches 0 in the weak topology \( L(y) \leq L(0) \), which proves that \( L \) is upper semicontinuous at 0 in the weak topology on \( S \). Continuity of \( L \) at 0 follows from the lower semicontinuity of the supremum of continuous functions and the definition of \( L \). For any other point \( y \in S \) appropriate translations reduce the problem to the case just considered where \( y = 0 \) and \( L(y) = 0 \).

Since \( F \) is \( J \)-monotone, for each \( u_j \) there is a \( z_j \in F(x) \) such that \( (z_j - u_j, J(x - u_j)) \geq 0 \). Thus, for each \( j \geq i \), \( \sup_{y \in B_j} L_j(y) \geq 0 \). The maximum theorem in [1] implies that
\[
\lim_{J \rightarrow \infty} \sup_{y \in B_0} L_i(y) = \sup_{y \in B_0} L_i(y) \geq 0 \quad \text{for } i = 1, 2, \ldots.
\]

For each positive integer \(i\) let \(A_i = \{ y \in B_0 : L_i(y) \geq -1/i \}\). If \(y \in A_{i+1}\) then, by the definition of the \(L_i\), \(L_i(y) \geq L_{i+1}(y) > -1/(i+1) > -1/i\), and therefore, \(y \in A_i\). Thus, \(A_{i+1} \subseteq A_i\) and, since the \(A_i\) are weakly closed subsets of the weakly compact set \(B_0\), there is a point \(y_0 \in \bigcap_{i=1}^{\infty} A_i\). By the definition of \(B_0\), \(y_0 = z_0 - v_0\) where \(z_0 \in F(x)\), and by (5) for some subsequence \(\{u_k\}\) of \(\{u_i\}\), \((z_0 - v_0, J(x - u_k)) \geq -1/k\). Taking the limit in the latter inequality we have \((z_0 - v_0, J(x - u_0)) \geq 0\), which contradicts (4).

**Theorem 1.** Let \(B\) be a separable strictly convex reflexive Banach space with a weakly continuous duality mapping \(J\), and \(C\) a closed bounded convex subset of \(B\). If \(G\) is a nonexpansive mapping of \(B\) into \(K(B)\) with the Hausdorff metric \(D\), which maps \(C\) into itself, then \(G\) has a fixed point in \(C\).

**Proof.** Assume without loss of generality that \(0 \in C\) [5]. The proof consists of showing that 0 is in the closure of \((I - G)(C)\) and that \(I - G\) is \(J\)-monotone. The theorem then follows by Lemma 2.

Let \(\{k_i\} \subseteq [0,1)\) be a sequence which converges to 1, and consider the sequence of mappings \(\{k_iG\}\) of \(C\) into \(K(C)\). By a result of Nadler [8] each \(k_iG\) has a fixed point \(x_{k_i} \in C\). Since \(x_{k_i} \in k_iG(x_{k_i})\) we have \(x_{k_i} = k_iy_{k_i}\), where \(y_{k_i} \in G(x_{k_i})\). Therefore,

\[
\inf_{y \in G(x_{k_i})} \|y - x_{k_i}\| \leq \|x_{k_i} - y_{k_i}\| \leq (1 - k_i) \|y_{k_i}\|
\]

and the last term tends to 0. This shows that 0 is in the closure of \((I - G)(C)\). \(G\) being nonexpansive with closed convex values, given any \(y \in B\) and \(y_1 \in G(y)\), there is a closest point \(x_1 \in G(x)\) to \(y_1\) such that \(\|x_1 - y_1\| \leq \|x - y\|\). It follows that

\[
((x - x_1) - (y - y_1), J(x - y)) \geq (\|x - y\| - \|x_1 - y_1\|) \|J(x - y)\| \geq 0,
\]

and hence \(I - G\) is \(J\)-monotone. Applying Lemma 2 we have \(0 \in (I - G)\cdot(C)\); i.e., there is an \(x \in C\) such that \(x \in G(x)\).

**Theorem 2.** Let \(B\) be a separable strictly convex reflexive Banach space with a weakly continuous duality mapping \(J\), and \(C\) a closed bounded convex subset of \(B\). Assume that \(\{G_i\}\) is a sequence of nonexpansive mappings of \(B\) into \(K(B)\) with the Hausdorff metric, which converges pointwise to a nonexpansive mapping \(G_0\) and maps \(C\) into itself. If \(x_i \in C\) is a fixed point of \(G_i\), \(i = 1, 2, \ldots\), and \(x_i \rightarrow x_0\) then \(x_0\) is a fixed point of \(G_0\).

**Proof.** As in the proof of Theorem 1 the mappings \(I - G_i, i = 0, 1, \ldots\), are \(J\)-monotone. Since \(x_i\) is a fixed point of \(G_i\), \(0 \in (I - G_i)(x_i)\) for \(i = 1, 2, \ldots\), and by \(J\)-monotonicity for each \(v \in B\) there is a \(v_i \in (I - G_i)(v)\) for which

\[
(v_i - 0, J(v - x_i)) \geq 0, \quad i = 1, 2, \ldots.
\]
Define the sequence of functionals \( \{L_i\} \) on \( B \) by

\[
L_i(y) = \sup_{j \geq i} (y, J(v - x_j)).
\]

As in the proof of Lemma 2 the \( \{L_i\} \) are continuous on \( S \) with the weak topology. Defining the sequence of weakly compact convex subsets \( \{B_j\} \) of \( B \) by \( B_j = (I - G_j)(v) \), the pointwise convergence of the \( \{G_j\} \) implies that \( \{B_j\} \) converges to \( B_0 = (I - G_0)(v) \) in the Hausdorff metric \( D \), and therefore, as shown in the proof of Lemma 2, the \( \{B_j\} \) converge to \( B_0 \) in the Hausdorff metric \( H \) generated by the weak metric topology on any ball containing the \( \{B_j\} \). Inequality (6) implies that for each \( i \), \( \sup_{y \in B_j} L_i(y) > 0 \) for \( j \geq i \).

Thus, the sequences \( \{L_i\} \) and \( \{B_j\} \) satisfy the same conditions as in the proof of Lemma 2, and therefore there is a point \( y_0 \in B_0 \) such that \( (y_0 - 0, J(v - x_0)) > 0 \). The point \( v \in B \) was arbitrary and so by Lemma 1, \( 0 \in (I - G_0)(x_0) \) and \( x_0 \) is a fixed point of \( G_0 \).

**References**


