A NEW FLAG TRANSITIVE AFFINE PLANE OF ORDER 27

M. L. NARAYANA RAO AND K. KUPPUSWAMY RAO

Abstract. A flag transitive affine plane of order 27 is constructed. The stabilizer of the origin of this plane contains a cyclic subgroup which is transitive on the lines through the origin. It is also shown that this plane is not isomorphic to the flag transitive plane constructed by Hering.

1. Let \( \pi \) be a finite affine plane of order \( n \). A collineation group \( G \) of \( \pi \) is defined to be flag transitive on \( \pi \) if \( G \) is transitive on incident point-line pairs or flags of \( \pi \). A. Wagner [13] has shown that \( \pi \) is a translation plane so that \( n = p^r \) for some prime \( p \) and for some integer \( r > 0 \). D. A. Foulser [4], [5] has determined all flag transitive groups of finite affine planes. While determining the flag transitive groups, Foulser remarks that the existence of non-Desarguesian flag transitive affine planes is still an open problem. However, Foulser constructs two flag transitive planes of order 25 [4] and shows that his two planes and the near field plane of order 9 have flag transitive collineation groups. C. Hering [7] has constructed a plane of order 27 which has a flag transitive collineation group. Recently, one of the authors [10], [11] has constructed a flag transitive plane of order 49 and a class of flag transitive planes of order \( q^2 \) where \( q \) is power of a prime \( p > 3 \). The aim of this paper is to construct a non-Desarguesian flag transitive plane of order 27 and establish that this is different from the plane constructed by Hering [7].

2. Let \( F \) be the set of all ordered triples \( (a, b, c) \) over \( GF(3) \). Let \( C \) be a set of \( 3 \times 3 \) matrices over \( GF(3) \) satisfying:

\[
\begin{align*}
(2.1) \quad & (i) \ C \text{ contains } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
(ii) \ C \text{ contains } 27 \text{ matrices,} \\
(iii) \text{ If } M, N \in C \text{ and } M \neq N, \text{ then } |M - N| \neq 0, \text{ where } |X| \text{ denotes determinant of matrix } X.
\end{align*}
\]

The conditions (2.1) imply that corresponding to each ordered triple \( (a, b, c) \) in \( F \), there is exactly one matrix of the form

Received by the editors October 17, 1974.
Key words and phrases. Projective planes, affine planes, flag transitivity, collineations.
which will be denoted by $M(a,b,c)$. We now define addition $\cdot$ and multiplication $\cdot$ on $F$ as follows.

\[
\begin{align*}
(a, b, c) + (d, e, f) &= (a + d, b + e, c + f) , \\
(a, b, c) \cdot (d, e, f) &= (d, e, f) M(a, b, c)
\end{align*}
\]

where $M(a, b, c)$ is the matrix in $C$ corresponding to $(a, b, c)$.

**Theorem 2.1.** The set $F$ with operations $\cdot$ and $\cdot$ defined by (2.2) is a left Veblen-Wedderburn system.

**Proof.** See [3, §5] or [2, §5].

3. We now construct a set $C$ of matrices over $GF(3)$ and show that $C$ satisfies conditions (2.1).

Let $T = \begin{pmatrix} 0 & P & Q & R \end{pmatrix}$ be a $6 \times 6$ matrix over $GF(3)$ where $0$ is $3 \times 3$ zero matrix and $P, Q, R$ are $3 \times 3$ nonsingular matrices given by

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 2 & 1 \\
2 & 2 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 2 & 1 \\
0 & 0 & 1 \\
1 & 2 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 2 & 1 \\
0 & 0 & 2 \\
2 & 0 & 2
\end{pmatrix}.
\]

Let

\[
M_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Matrices $M_i$, $2 \leq i \leq 27$, are inductively defined as

\[
M_2 = Q^{-1} R = \begin{pmatrix}
1 & 1 & 0 \\
2 & 1 & 1 \\
0 & 0 & 2
\end{pmatrix}, \quad
M_{k+1} = Q^{-1} M_k^{-1} P + M_2, \quad 2 \leq k \leq 26.
\]

An inspection of Table 3.1 shows that for each $k$, $3 \leq k \leq 26$, $M_{k-1}$ is nonsingular and, therefore, $M_k$ is defined for $3 \leq k \leq 27$.

**Lemma 3.1.** The set $C = \{M_i | i = 0, 2 \leq i \leq 27\}$ satisfies the conditions (2.1) and, hence, the corresponding $F(\cdot, \cdot)$ is a left Veblen-Wedderburn system.

**Proof.** An inspection of Table 3.1 shows that $C$ contains the zero and the unit matrices, the nonzero matrices of $C$ are nonsingular and $C$ consists of 27 matrices in all. Further, $Q^{-1} M_2^{-1} P + Q^{-1} R = M_0$. From (3.1) we obtain

\[
|M_{k+1} - M_2| = |Q^{-1}||M_{k-1}||R| \neq 0
\]

since $Q, R$ and $M_0$ are nonsingular for $2 \leq k \leq 26$. Since $M_0$ is the $3 \times 3$ zero matrix and $M_k$ is nonsingular for $2 \leq k \leq 27$, 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $1 < k < 26$, $1 < j < 26$, $k \neq j$. There is no loss of generality if we assume $k > j$. From (3.1) we obtain

$$M_{k+1} - M_j = Q^{-1} M_{k}^{-1} (M_j - M_k) M_j^{-1} R$$

which implies that

$$|M_{k+1} - M_j| = \mu(j, k) |M_j - M_k|$$

(3.4)

where $\mu(j, k) = |Q^{-1} M_{k}^{-1} (M_j - M_k) M_j^{-1} R| \in GF(3)$ and $\mu(j, k) \neq 0$. Applying (3.4) repeatedly we obtain

$$|M_{k+1} - M_j| = \mu |M_{k-j+2} - M_2|$$

(3.5)

where $\mu$ is a nonzero element of $GF(3)$ which depends upon $k$ and $j$. In view

| $M_k - M_0$ | $\neq 0$. |
|--------------|

<table>
<thead>
<tr>
<th>$i$</th>
<th>$M_i$</th>
<th>$i$</th>
<th>$M_i$</th>
<th>$i$</th>
<th>$M_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0</td>
<td>10</td>
<td>0 1 0 2</td>
<td>19</td>
<td>1 1 0</td>
</tr>
<tr>
<td>0</td>
<td>0 0 0</td>
<td>11</td>
<td>1 2 1</td>
<td>12</td>
<td>1 2 2</td>
</tr>
<tr>
<td></td>
<td>1 1 0</td>
<td>0 2 2</td>
<td>0 1 0</td>
<td>1 1 0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2 1 1</td>
<td>12</td>
<td>0 2 0</td>
<td>21</td>
<td>1 1 0</td>
</tr>
<tr>
<td></td>
<td>0 0 2</td>
<td>2 0 2</td>
<td>0 2 2</td>
<td>2 1 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0 1</td>
<td>2 1 2</td>
<td>2 1 2</td>
<td>2 2 1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 2 2</td>
<td>13</td>
<td>0 2 0</td>
<td>22</td>
<td>2 1 0</td>
</tr>
<tr>
<td></td>
<td>1 1 2</td>
<td>2 0 1</td>
<td>1 1 2</td>
<td>2 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 2 0</td>
<td>0 1 2</td>
<td>2 2 1</td>
<td>2 2 0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2 2 0</td>
<td>14</td>
<td>0 1 0</td>
<td>23</td>
<td>2 0 2</td>
</tr>
<tr>
<td></td>
<td>1 2 0</td>
<td>0 0 1</td>
<td>1 2 0</td>
<td>2 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 2 0</td>
<td>0 1 0</td>
<td>2 0 0</td>
<td>1 2 0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0 2 1</td>
<td>15</td>
<td>0 2 0</td>
<td>24</td>
<td>1 0 0</td>
</tr>
<tr>
<td></td>
<td>2 1 0</td>
<td>1 2 1</td>
<td>1 2 0</td>
<td>1 2 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 0</td>
<td>0 2 0</td>
<td>0 1 0</td>
<td>0 2 1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1 0 1</td>
<td>16</td>
<td>0 2 2</td>
<td>25</td>
<td>1 1 1</td>
</tr>
<tr>
<td></td>
<td>0 2 1</td>
<td>1 1 2</td>
<td>2 2 0</td>
<td>1 1 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 0 2</td>
<td>0 0 1</td>
<td>2 0 2</td>
<td>1 1 1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0 0 2</td>
<td>17</td>
<td>0 1 2</td>
<td>26</td>
<td>0 0 1</td>
</tr>
<tr>
<td></td>
<td>1 1 0</td>
<td>2 2 1</td>
<td>2 0 2</td>
<td>1 0 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 2 2</td>
<td>2 0 1</td>
<td>2 2 2</td>
<td>1 0 2</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0 1 1</td>
<td>18</td>
<td>2 2 1</td>
<td>27</td>
<td>2 1 2</td>
</tr>
<tr>
<td></td>
<td>0 1 2</td>
<td>1 0 1</td>
<td>2 1 2</td>
<td>2 1 1</td>
<td></td>
</tr>
</tbody>
</table>
of (3.2) we get

\[ |M_{k+1} - M_{j+1}| \neq 0. \]

Now the lemma follows from (3.2), (3.3), (3.6) and Theorem 2.1.

4. Let \( \Pi_N \) be the projective plane coordinatized by \( F(+, \cdot) \) [4, p. 353], [6, p. 45], [10, p. 31]. That is \( \Pi_N \) has points \((c), (a,d), (\infty)\) and lines \([k], [m,b], [\infty] = L_\infty \) for \( a, b, c, d, k, m \in F \) and \( \infty \notin F \). Incidence in \( \Pi_N \) is defined by \((x,y) \in [m,b] \) if \( y = m \cdot x + b \) and \((x,y) \in [k] \) if \( x = k \). The plane \( \Pi_N \) may also be considered as a six dimensional right vector space \( V(6,3) \) with three dimensional subspaces of \( V(6,3) \) as lines and vectors of \( V(6,3) \) as points. The line corresponding to the equation \( y = m \cdot x, m \in F \), is given by the subspace

\[ V(m) = \{(a,b,c,d,e,f) \mid (a,b,c) \in F, (d,e,f) = (a,b,c)M(m)\} \]

where \( M(m) \) is the unique matrix in \( C \) corresponding to \( m \in F \). The line \( x = 0 \) corresponds to the subspace

\[ V(\infty) = \{(0,0,0,a,b,c) \mid (a,b,c) \in F\}. \]

The lines \( y = m \cdot x + b \) correspond to appropriate translates of \( V(m) \) for \( m \in F \) or \( m = \infty \). The group \( G_0 \) of all collineations fixing \((0,0,0)\) of \( \Pi_N \) consists of all nonsingular linear transformations of \( V(6,3) \) which permute the subspaces \( V(m) \) for \( m \in F \) or \( m = \infty \) among themselves [1, Satz 19], [12, p. 208].

We now show that \( T \) given in §3 is a collineation of \( \Pi_N \) fixing the point corresponding to zero vector in \( V(6,3) \). For convenience let the subspace \( V(m) \) be denoted by \( V_k \), where \( M_k \) is the unique matrix in \( C \) corresponding to \( m \in F \), and \( V(\infty) \) be denoted by \( V_1 \). It is clear that \( T \) is nonsingular. Since \( P \) is nonsingular it is easy to see that \( V_0 T = V_1 \). Now

\[ V_1 T = \{(0,0,0,a,b,c) \mid (a,b,c) \in GF(3)\} T \]

\[ = \{(x,y,z,p,q,r) \mid (x,y,z) = (a,b,c)Q \]

and \( (p,q,r) = (a,b,c)R, (a,b,c) \in GF(3)\} \]

\[ = \{(x,y,z,p,q,r) \mid (p,q,r) = (x,y,z)Q^{-1}R \]

\[ = (x,y,z)M_2, (x,y,z) \in GF(3)\} \]

\[ = V_2. \]

Similarly, in view of relations

\[ M_{i+1} = Q^{-1} M_i^{-1} P + Q^{-1} R \quad \text{and} \quad M_0 = Q^{-1} M_{27}^{-1} P + Q^{-1} R, \]

we obtain that \( V_i T = V_k \) where \( k = i + 1 \) (mod 28) for \( 2 \leq i \leq 27 \). Applying \( T \) repeatedly we obtain that

\[ V_i T^j = V_k \quad \text{where} \quad k = i + j \pmod{28}. \]

Thus \( \langle T \rangle \) is a collineation group of \( \Pi_N \) which permutes the lines through the
origin transitively. The action of $T$ restricted to the lines through the origin of $\Pi_N$ may be given by $T: (0, 1, 2, 3, \ldots, 27)$ where $x$ stands for the line $V_x$. Let $\pi_N$ be the affine plane obtained from $\Pi_N$ by deleting the line containing the ideal points.

**Theorem 4.1.** The plane $\pi_N$ is a non-Desarguesian flag transitive affine plane.

**Proof.** An examination of Table 3.1 reveals that the set $C = \{M_i|i = 0, 2 \leq i \leq 27\}$ of matrices does not form a Galois field under matrix addition and multiplication implying $\Pi_N$ is non-Desarguesian [3, p. 230]. Thus $\pi_N$ is non-Desarguesian. The rest of the theorem follows from the action of $T$ restricted to the lines of $\pi_N$ through the origin.

**Lemma 4.2.** Any collineation of $\pi_N$ fixing $V_0, V_1, V_{14}, V_{15}$ and permuting the other lines through the origin among themselves fixes $V_7$ also.

**Proof.** Let $S = (U^T V X)$, where $U, V, W, X$ are $3 \times 3$ matrices over $GF(3)$, be a matrix inducing a collineation of $\pi_N$ fixing $V_0, V_1, V_{14}, V_{15}$ and permuting the other lines of $\pi_N$ through the origin. Since $S$ fixes $V_0$ and $V_1$, $S$ reduces to the form $(U^T 0 X)$ where $0$ is $3 \times 3$ zero matrix, and $U$ and $X$ are $3 \times 3$ nonsingular matrices [9, Lemma 1.2]. Since $S$ permutes the lines $V_i, 2 \leq i \leq 27$, among themselves, $U$ and $X$ must be such that for each $M_i \in C$ there must be a $M_j \in C$ satisfying $UM_i = M_j X, i \neq 0, 1, J \neq 0, 1$. Obviously, $S$ fixes $V_j$ if and only if $UM_i = M_j X$. Since $S$ fixes $M_{14}$ and $M_{14} = I, the 3 \times 3$ unit matrix, we get that $U = X$. But $S$ fixes $V_{15}$ also so that $UM_{15} U^{-1} = M_{15}$. An examination of Table 3.1 shows that

$$M_7 = M_{14} + M_{15} = I + M_{15}.$$  

Then

$$UM_7 U^{-1} = UIU^{-1} + UM_{15} U^{-1} = I + M_{15} = M_7.$$  

Thus $S$ fixes $V_7$ also. Hence, the lemma.

5. We give a brief description of the construction of Hering's affine plane $\pi_H$ [7] and deduce some of its collineations in order to compare it with the plane $\pi_N$. Let $0$ be $3 \times 3$ zero matrix. Let

$$s = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{where} \quad A_1 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$r = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \quad \text{where} \quad B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix} \quad \text{where} \quad C_1 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$
Let $L_{26}$ and $L_{27}$ be three dimensional subspaces of $V(6,3)$ defined by basis vectors as

\[ L_{26} = \langle (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0) \rangle, \]
\[ L_{27} = \langle (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1) \rangle. \]

Let

\[ L_i = L_{26} s^i, \quad 0 \leq i \leq 12, \quad L_{13+i} = L_{27} s^i, \quad 0 \leq i \leq 12. \]

The incidence structure $\pi_H$ with $L_i$ ($0 \leq i \leq 27$) and their cosets in the additive group of $V(6,3)$ as lines and vectors of $V(6,3)$ as points with inclusion as incidence relation is the flag transitive plane of order 27 constructed by Hering [7].

We will now show that $s, h$ and $r$ induce collineations on $\pi_H$. It is easy to verify that

\[ L_{26} s = L_{26} \quad \text{and} \quad L_{27} s = L_{27} \]
\[ L_i s^i = L_k \quad \text{where} \quad k \equiv i + j \pmod{13}, \quad 0 \leq i \leq 12, \quad 0 \leq j \leq 12, \quad \text{and} \]
\[ L_{13+i} s^i = L_{13+k} \quad \text{where} \quad k \equiv i + j \pmod{13}. \]

Thus $s$ induces a collineation on $\pi_H$ the action of which restricted to lines through the origin is given by $s$: $(26)(27)(0,1,2,\ldots,12)(13,14,\ldots,25)$. Here $(x)$ stands for line $L_x$.

In order to consider actions of $h$ and $r$ on $\pi_H$ we need the following relations between generators $r, h$ and $s$ of the group $G$ considered by Hering [7].

\[ s^{13} = I, \quad h^6 = r^2 = 2I, \quad r^{-1} hr = h^{-1}, \]

where $I$ is the $6 \times 6$ unit matrix.

\[ h^{-1} sh = s^4, \quad r^{-1} sr = s^{-1} r s^{-1}. \]

For $1 < i < 12$, $r^{-1} s^i r$ can be computed according to the relations given by Hering [7, p. 205].

It can be checked that

\[ L_{26} h = L_{27}, \quad L_{27} h = L_{26} \quad \text{and} \quad L_0 h = L_{26} rh = L_{26} h^{-1} r = L_{27} r = L_{13}. \]

Similarly, $L_{13} h = L_0$. Let $1 < i < 12$. Then

\[ L_i h = L_{26} r s^i h = L_{26} rh h^{-1} s^i h = L_{26} r h s^4 i \]
\[ = L_{26} h^{-1} r s^4 i = L_{27} r s^4 i = L_{13} s^4 i = L_{13+(4i \pmod{13})}. \]

Similarly, $L_{13+i} h = L_{4i \pmod{13}}$.

Thus it follows that $h$ induces a collineation and its action restricted to the lines of $\pi_H$ through origin is

\[ h: (26,27), (0,13), (1,17,3,25,9,23), (2,21,6,24,5,20), \]
\[ (4,16,12,10,14), (7,15,8,19,11,18). \]

The action of $h^2$ restricted to the lines $\pi_H$ is given by
A NEW FLAG TRANSITIVE AFFINE PLANE OF ORDER 27

\[ h^2: (26)(27)(0)(13)(1,3,9)(2,6,5)(4,12,10)(7,8,11)(14,16,22) \]
\[ (15,19,18)(17,25,23)(20,21,24). \]

From the definition of \( L_0 \) and \( L_{13} \) we get \( L_{26}r = L_0 \) and \( L_{27}r = L_{13} \). Since \( r^2 = 2I \), \( L_0r = L_{26}r^2 = L_{26} \). Similarly \( L_{13}r = L_{27} \). From \( r^{-1}s = s^2rs^2 \) it follows that

\[ L_1r = L_{26}rsr = L_{26}r^2r^{-1}s = L_{26}s^{12}rs^{12} = L_{12}. \]

Here we have used the facts that \( L_{26}r^2 = L_{26} \) and \( L_{26}s^i = L_{26} \) for any \( i \).
Similarly, using conjugates of \( s^i \) by \( r \) and noting that \( h \) interchanges \( L_{26} \) and \( L_{27} \), we may conclude that \( R \) induces a collineation and its action restricted to the lines \( L_i \) is

\[ r: (0,26)(13,27)(1,12)(2,19)(3,4)(5,18)(6,15)(7,24)(8,21)(9,10) \]
\[ (11,20)(14,25)(16,17)(22,23). \]

**Theorem 5.1.** Any collineation of \( \pi_H \) that fixes \( L_{26} \) also fixes \( L_{27} \).

**Proof.** Let \( \beta \) be any collineation of \( \pi_H \) fixing \( L_{26} \) and moving \( L_{27} \) to \( L_x \), \( 0 \leq x \leq 12 \). Since \( s \) fixes \( L_{26} \) and is transitive on the lines \( L_i \), \( 0 \leq i \leq 12 \), we may as well take \( \beta \) such that \( \beta \) fixes \( L_{26} \) and maps \( L_{27} \) onto \( L_0 \). Then the action of

\[ \beta^{-1}s\beta = (26)(0)(x_0,x_1,...,x_{12})(y_{13},...,y_{25}), \]

where \( L_x = L_\beta; L_{y_{13+i}} = L_{13+i} \beta \). Now two cases need to be considered.

**Case (i).** Suppose at least one number \( i, 0 \leq i \leq 12 \), and at least one number \( j, 13 \leq j \leq 25 \), are contained in the set \( \{x_0,...,x_{12}\} \). Then obviously \( G_1 = \langle s, \beta^{-1}s \rangle \) fixes \( L_{26} \) and is transitive on the remaining lines through the origin. Then \( \langle G, G_1 \rangle \) is doubly transitive on the lines of \( \pi_H \) through the origin.

**Case (ii).** Suppose \( \{x_0,...,x_{12}\} \) consists of the numbers \( \{13,14,...,25\} \). Then the action of \( r^{-1}\beta^{-1}s\beta r \) to the lines \( L_i \) of \( \pi_H \) through the origin is given by

\[ r^{-1}\beta^{-1}s\beta r: (26)(0)(z_0,z_1,...,z_{12})(a_0,a_1,...,a_{12}) \]

where \( L_{z_j} = L_{13+i}r \) for some \( j, 0 \leq j \leq 12 \).

Since \( L_{19}r = L_2, L_{17}r = L_{16}, L_{13}r = L_{27} \) we find that \( \{z_0,z_1,...,z_{12}\} \) contains 2,16 and 27. This implies that \( G_2 = \langle s, r^{-1}\beta^{-1}s\beta r \rangle \) fixes \( L_{26} \) and is transitive on the remaining lines of \( \pi_H \) through the origin. Here again, the group \( \langle G, G_2 \rangle \) is doubly transitive on the lines of \( \pi_H \) through the origin.

Similarly we may conclude that the collineation group of \( \pi_H \) is doubly transitive on the lines of \( \pi_H \) through the origin if \( \beta \) fixes \( L_{26} \) and maps \( L_{27} \) onto \( L_{13} \).

Since the order 27 of \( \pi_H \) is odd and 27 \( \equiv 1 \) (mod 8), the double transitivity implies that \( \pi_H \) is Desarguesian [3, p. 217]. This is a contradiction since \( \pi_H \) is non-Desarguesian. From this contradiction the truth of lemma follows.

**Definition 5.2.** If every collineation of \( \pi_H \) fixing a line \( L_x \) also fixes \( L_y \), then \( L_y \) is called a companion of \( L_x \).
Lemma 5.3. Every line $L_x$ of $\pi_H$ through the origin has a unique companion.

Proof. By Theorem 5.1 any collineation that fixes $L_{26}$ also fixes $L_{27}$ and $L_{27}$ is, therefore, a companion of $L_{26}$. Since $s$ fixes $L_{26}$ and $L_{27}$ and moves the remaining lines through the origin, none of the lines through the origin other than $L_{27}$ can be a companion of $L_{26}$. Thus $L_{27}$ is the unique companion of $L_{26}$. From the transitivity of $G$ on the lines $L_i$, we get that every line $L_x$ has a unique companion $L_y$. We may also remark that if $L_y$ is a companion of $L_x$ then $L_x$ is a companion of $L_y$.

Lemma 5.4. $\pi_H$ has a collineation the restriction of which to the lines $L_i$ fixes each line of any two pairs of companions and moves the remaining lines through the origin.

Proof. Follows from Lemma 5.3, structure of $h^2$ and $s$ and the transitivity of $G$ on the lines $L_i$.

Lemma 5.5. If $\pi_N$ is Hering’s plane, then the line $V_k$ is the unique companion of $V_x$ where $k \equiv x + 14 \pmod{28}$.

Proof. Suppose $\pi_N$ is Hering’s plane. Then there is a collineation $\gamma$ whose action restricted to the lines $V_i$ is given by

$$\gamma: (x)(y)(a_0, a_1, \ldots, a_{12})(b_0, b_1, \ldots, b_{12})$$

where $V_y$ is the unique companion of $V_x$. Then $T^{y-x}\gamma T^{x-y}$ fixes $V_x$ and $V_y$ where $j \equiv 2x - y \pmod{28}$, moving the remaining lines. In view of Lemma 5.3, $V_y$ must be $V_x$. This is possible only when $y \equiv 2x - y \pmod{28}$ which gives $2x \equiv 2y \pmod{28}$, since $x \neq y$. This implies $y = x + 14 \pmod{28}$.

Theorem 5.6. $\pi_N$ is not isomorphic to $\pi_H$.

Proof. If $\pi_N$ is isomorphic to $\pi_H$, then in view of Lemmas 5.4 and 5.5, $\pi_N$ has a collineation fixing $V_0$, $V_{14}$, $V_i$ and $V_{15}$ and moving the remaining lines through the origin. But this is not possible by Lemma 4.2. Hence $\pi_N$ is not isomorphic to $\pi_H$.

References


DEPARTMENT OF MATHEMATICS, OSMANIA UNIVERSITY, HYDERABAD-500007, INDIA