PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, September 1976

PROJECTIONS FROM $L^p(G)$ ONTO CENTRAL FUNCTIONS

BENJAMIN B. WELLS, JR.

ABSTRACT. Let G denote a locally compact $[FC]_B^-$-group. Then for every $1 \leqslant p \leqslant \infty$ there is a projection from $L^p(G)$ onto the B-central functions in $L^p(G)$.

Let G denote a locally compact group and B a subgroup of the group $\mathrm{Aut}(G)$ of topological automorphisms of G. We shall assume that B contains all inner automorphisms and that each element of B is Haar-measure-preserving. For $1 \leqslant p \leqslant \infty$ the B-center of $L^p(G)$, $Z^B(L^p(G))$, is defined to be the set of all $f \in L^p(G)$ such that $f^\beta = f$ for all $\beta \in B$ (where $f^\beta(x) = f(\beta^{-1}x)$). For $p = 1$, $Z^B(L^1(G))$ is a central Banach subalgebra of $L^1(G)$ (see [8, p. 148]). The group G is an $[FC]_B^-$-group if G has precompact B-orbits. By a result of Liukkonen [7, p. 90] any such group has a compact B-invariant neighborhood of the identity, i.e. is an $[IN]_B$-group.

THEOREM. *Let G be a locally compact $[FC]_B^-$-group. Then for every $1 \leqslant p \leqslant \infty$ there is a projection (norm 1, linear, idempotent) P: $L^p(G) \to Z^B(L^p(G))$. In addition, P commutes with the operation $x \to x^{-1}$ and preserves positive functions. If $p = 1$, then*

$$P(Pf * g) = Pf * Pg \quad \textit{and} \quad P(f * g) = P(g * f).$$

If $p = 2$, then P is an orthogonal projection.

This result is essentially known (cf. [6] and [8]). [3, Theorem 2.1, p. 424] contains a version of it in the context of central topological groups. [10, Propositions 1.3–1.5] contains a version for $[FIA]_B^-$-groups. All of the constructions of the projection operators described in the literature involve integration over the compact group B^-. The purpose of this note is to offer a different and perhaps more natural construction of the projections of the theorem and to point out that they coincide with those already in the literature.

SKETCH OF THE PROOF OF THE THEOREM. Since G is an $[IN]$-group, $Z^B(L^1(G)) \neq (0)$, and by a result of [2] G is unimodular. Let m denote Haar measure on G and Γ the sigma-field of B-invariant Borel sets. Since G is

Received by the editors July 7, 1975 and, in revised form, January 12, 1976.

AMS (MOS) subject classifications (1970). Primary 43A15.

Key words and phrases. Locally compact $[FC]_B^-$-group, B-invariant Borel set, B-central function, projection.

$[FC]_B^-$, the measure space (G, Γ, m) is decomposable (see [5, 19.25–19.27]). For a nonnegative locally integrable Haar-measurable function f, the Radon-Nikodym Theorem [5, 19.27] may be applied to get a Γ-measurable function $E(f)$, called the conditional expectation of f with respect to the sigma-field Γ. One then extends E to complex-valued locally integrable functions in the usual way. Now take $P(f) = E(f)$. It is straightforward to verify that P has the properties mentioned in the Theorem.

Furthermore, any two projections with the properties of the Theorem necessarily coincide on all $L^p(G)$ $(1 \leqslant p \leqslant \infty)$ since they are each equal to the orthogonal projection of $L^2(G)$ onto $Z^B(L^2(G))$, hence agree on the continuous functions with compact support, and thus on all of $L^p(G)$.

Finally, the author would like to thank Professor Kenneth Ross for bringing to his attention the papers [8]–[10] and making several helpful suggestions that improved the statement of the Theorem.

References

1. S. Grosser and M. Moskowitz, *On central topological groups*, Trans. Amer. Math. Soc. **127** (1967), 317–340. MR **35** #292.

2. ———, *Representation theory of central topological groups*, Trans. Amer. Math. Soc. **129** (1967), 361–390. MR **37** #5327

3. ———, *Harmonic analysis on central topological groups*, Trans. Amer. Math. Soc. **156** (1971), 419–454. MR **43** #2165.

4. E. Hewitt and K. Ross, *Abstract harmonic analysis*. Vol. I, Springer-Verlag, Berlin; Academic Press, New York, 1963. MR **28** #158.

5. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1965. MR **32** #5826.

6. E. Kaniuth and G. Schlichting, *Zur harmonischen Analyse klassenkompakter Gruppen*. II, Invent. Math. **10** (1970), 332–345. MR **47** #5521.

7. J. Liukkonen, *Dual spaces of groups with precompact conjugacy classes*, Trans. Amer. Math. Soc. **180** (1973), 85–108. MR **47** #6937.

8. J. Liukkonen and R. Mosak, *Harmonic analysis and centers of group algebras*, Trans. Amer. Math. Soc. **195** (1974), 147–163. MR **50** #2815.

9. R. D. Mosak, *Central functions in group algebras*, Proc. Amer. Math. Soc. **29** (1971), 613–616. MR **43** #5323.

10. ———, *The L^1- and C^*-algebras of $[FIA]_B^-$-groups, and their representations*, Trans. Amer. Math. Soc. **163** (1972), 277–310. MR **45** #2096.

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822