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CONTRACTIVE PROJECTIONS IN SQUARE BANACH SPACES

NINA M. ROY

ABSTRACT. It is proved that if X is a square space and P is a contractive projection in X, then PX is square; and if X is regular, then PX is regular. It is also shown that a regular square space is isometric to the image, under a contractive projection, of a regular (square) Kakutani M-space. These results are analogous to those obtained for other classes of L_1-preduals by Lindenstrauss and Wulbert, and in this paper their diagram of L_1-preduals is enlarged so as to include the classes of square, regular square and regular M-spaces.

A square space is a real Banach space which is (linearly) isometric to a uniformly closed linear space X of real functions on a compact Hausdorff space Ω satisfying:

(i) $|x|$ is upper semicontinuous for each x in X, and

(ii) X is invariant under multiplication by functions in $C(\Omega)$.

Square spaces were introduced in [3] and characterized in [8] in terms of the (Alfsen-Effros) structure topology on the set E of extreme points of the dual ball. (See Lemma 1 below.) The class of square spaces, denoted here by S, is properly contained in the class G of Grothendieck G-spaces [3, Theorems 1, 2] as is the class M of Kakutani M-spaces. We call a G-space *regular* if the structure topology on E is regular, and we denote by M_r (respectively, S_r) the class of regular M-spaces (resp., regular square spaces).

If B is a class of Banach spaces, then, following the notation in [6] and [7], denote by $\pi(B)$ the class of all Banach spaces Y for which there exists an X in B with a subspace Y' isometric to Y and a contractive projection (i.e., a projection of norm 1) of X onto Y'. The main results in this paper are $\pi(S) = S$ (Theorem 1) and $\pi(M_r) = \pi(S_r) = S_r$ (Theorem 2).

The diagram below shows how the classes M_r, S_r, S and $M \cap S$ are related and fit into the diagram of Lindenstrauss spaces (L_1-preduals) in [7].

$$\begin{array}{ccccccccccc} C & \to & C_0 & \to & M_r & \to & M \cap S & \to & M & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ C_\Sigma & \to & C_\sigma & \to & S_r & \to & S & \to & G & \to & \{X\colon X^* = L_1(\mu)\} \end{array}$$

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The arrows indicate proper inclusion, and the intersection of two classes is their closest common source, e.g., $M \cap C_\sigma = C_0$. The reader is referred to [6] or [7] for precise definitions of the classes C, C_0, C_Σ, C_σ, M and G. (We are writing C in place of $C(K)$, C_0 in place of $C_0(K)$, etc.) It is proved in [7] that

$$\pi(C) = \pi(C_0) = \pi(C_\Sigma) = \pi(C_\sigma) = C_\sigma \quad \text{and} \quad \pi(M) = \pi(G) = G.$$

The (open) problem of characterizing $\pi(M \cap S)$ is discussed in the Remark at the end of this paper.

After the following preliminaries, we shall verify those parts of our diagram which are not in [7] (or [6]) and then proceed to our main results (Theorems 1 and 2).

In this paper, only real Banach spaces are considered, and we use the following notation and terminology. If X is a Banach space, then E (or, when necessary, E_X) denotes the set $\operatorname{ext} B(X^*)$ of extreme points of the closed unit ball $B(X^*)$ in the dual of X, and $\bar{E}$ is the weak* closure of E. We shall frequently regard X as a space of functions on E (or $\bar{E}$) and write $x(p)$ in place of $p(x)$, for x in X and p in E. We denote by $\mathfrak{M}(X)$ the algebra of multipliers of X; by definition, a bounded real function f on E belongs to $\mathfrak{M}(X)$ if for each x in X there is y in X satisfying $f(p)x(p) = y(p)$ for all p in E. (This algebra was introduced in [2].) The (Alfsen-Effros) structure topology on E is the topology whose nonempty closed sets are of the form $E \cap N$, where N is a nonzero weak* closed L-summand in X^* [1, Proposition 3.3]. Let $C_s^b(E)$ denote the algebra of all bounded, structurally continuous functions on E. An important result, used several times in this paper, is that $C_s^b(E) = \mathfrak{M}(X)$ [1, Theorem 4.9].

LEMMA 1 [8, COROLLARY 1]. *A Banach space X is square if and only if $C_s^b(E)$ separates linearly independent points in E.*

LEMMA 2 [8, LEMMA 5]. *If X is a G-space, then a base for the structure topology on E is the family $\{V_x\colon x \in X\}$, where $V_x = \{p \in E\colon x(p) \neq 0\}$.*

LEMMA 3. *If X is a G-space for which E is (structurally) regular, then E is completely regular.*

PROOF. Let $E_\sigma = \{\{p, -p\}\colon p \in E\}$ have the quotient structure topology. Then by [8, Theorem 2] and the fact that the closed sets in E are symmetric, we have the implications

$$E \text{ regular} \Rightarrow E_\sigma \text{ regular} \Rightarrow E_\sigma \text{ completely regular} \Rightarrow E \text{ completely regular.}$$

The following three propositions are useful in verifying the diagram.

PROPOSITION 1. $M_r = M \cap S_r$.

PROOF. It suffices to show that every regular G-space is square. Let X be a regular G-space. Then E is completely regular by Lemma 3. Since $\{p, -p\}$ is closed for each p in E (because X is a Lindenstrauss space) it follows from Lemma 1 that X is square.

The next two results follow immediately from the proofs of [8, Corollaries 2, 4] and the fact that if E_σ is completely regular, then so is E.

PROPOSITION 2. *Every separable G-space is in* S_r.

PROPOSITION 3. $C_\sigma \subset S_r$.

In view of the above propositions as well as results in [3] and [7], it is clear that to complete the verification of our diagram we need only show that those inclusions not appearing in [7] are all proper. Cunningham established [3] that $M \cap S \neq M$, $M \cap S \neq S$, and $S \neq G$; the example in [3, Theorem 3] together with Proposition 3 show that $M_r \neq S_r$. Effros constructed a separable M-space which is not a C_σ-space [5, Theorem 4.3]; hence $C_\sigma \neq S_r$ by Proposition 2 (also $C_0 \neq M_r$). Finally, a square space X which is not regular can be found in [8, pp. 147, 148]; that X is also an M-space can be verified as follows. By definition, X is the uniform closure of a linear space Y of functions (of the form $f + gx_1$, in the notation of [8]) on a compact ordinal space Ω. One first shows that if $y \in Y$, then for each t in Ω there is y_t in Y such that $|y(t)| = y_t(t)$ and $|y| - y_t$ is continuous at t. (Consider the three cases $t \in F$, $t \in L - F$, $t \notin L$.) Then the partition of unity technique used in [3, Proof of Theorem 1, p. 555] can be applied to show that if $y \in Y$, then $|y| \in X$. Therefore if $x \in X$, then $x^+ = (x + |x|)/2$ is in X, and so X is a closed linear sublattice of $m(\Omega)$, hence an M-space. This example proves that $M_r \neq M \cap S$ and $S_r \neq S$, hence the diagram is verified.

In what follows, $E = \operatorname{ext} B(X^*)$, $E_Y = \operatorname{ext} B(Y^*)$, and $E(K_q) = \operatorname{ext} K_q$.

THEOREM 1. $\pi(S) = S$.

PROOF. Let X be a square space and P a contractive projection on X. We shall use Lemma 1 to show that $Y = PX$ is square.

For each q in E_Y let K_q denote the set of all norm-preserving extensions of q in X^*. Then K_q is a convex weak* compact subset of $B(X^*)$, $P^*q \in K_q$, and $E(K_q) \subset E$. (P^* denotes the adjoint of P.) Further, $E(K_q)$ is weak* compact since X is square. For if $\{p_\alpha\}$ is a net in $E(K_q)$, then there is a subnet $\{p_\beta\}$ which converges weak* to some p in K_q. Then p is in $\overline{E} \cap K_q$, hence in $E \cap K_q$ because $\|p\| = 1$ and $\overline{E} \subset [0, 1]E$ [3, Lemma 2], [8, Lemma 2]. Thus $p \in E(K_q)$. It now follows that $E(K_q)$ is structurally compact since the structure topology on E is smaller than the (relativized) weak* topology on E.

Let q_1 and q_2 be linearly independent points in E_Y and let a and b be real numbers with $0 < a < b < 1$. Then there is f in $C_s^b(E)$ such that $f < a$ on $E(K_{q_1})$ and $f > b$ on $E(K_{q_2})$. To prove this, we use the "only if" part of Lemma 1 and the structural compactness of the $E(K_{q_i})$. If $s \in E(K_{q_2})$, then for each p in $E(K_{q_1})$ there is f_p in $C_s^b(E)$ with $f_p(p) = 0$ and $f_p(s) = 1$; let $U_p = \{r \in E(K_{q_1}): f_p(r) < a\}$. Then a finite number of these open sets, say $U_{p_1}, U_{p_2}, \ldots, U_{p_n}$, cover $E(K_{q_1})$. Let $f_s = f_{p_1} \wedge f_{p_2} \wedge \cdots \wedge f_{p_n}$. Then f_s

$\in C_s^b(E)$, $f_s < a$ on $E(K_{q_1})$, and $f_s(s) = 1$. For each s in $E(K_{q_2})$ choose such a function f_s and define $V_s = \{t \in E(K_{q_2})\colon f_s(t) > b\}$. Then $E(K_{q_2})$ is covered by a finite number of these open sets, say $V_{s_1}, V_{s_2}, \ldots, V_{s_m}$. Let $f = f_{s_1} \vee f_{s_2} \vee \cdots \vee f_{s_m}$. Then $f \in C_s^b(E)$, $f < a$ on $E(K_{q_1})$ and $f > b$ on $E(K_{q_2})$.

Let q_1, q_2, a, b and f be as above. We will show there is g in $C_s^b(E_Y)$ with $g(q_1) \leqslant a$ and $g(q_2) \geqslant b$, thus proving Y is square. For each q in E_Y we have that f is weak* continuous on the compact set $E(K_q)$; hence there is x_q in X such that $f = x_q$ on $E(K_q)$ and $\|x_q\| = \|f|E(K_q)\|$ [4, Lemma 6.1]. Define g on E_Y by $g(q) = x_q(P^* q)$ for each q in E_Y. For each p in $E(K_{q_1})$ we have $x_{q_1}(p) = f(p) < a$; hence $x_{q_1} \leqslant a$ on $K_{q_1} = \overline{\text{co}}\ E(K_{q_1})$, and so $g(q_1) = x_{q_1}(P^* q_1) \leqslant a$. Similarly $g(q_2) \geqslant b$. We now show g is in $\mathfrak{M}(Y) = C_s^b(E_Y)$. Clearly g is bounded, indeed $|g(q)| \leqslant \|f\|$ for each q in E_Y. Let $y \in Y$. Then, since $f \in \mathfrak{M}(X)$, there is x in X such that $fy = x$ on E. It follows that $gy = Px$ on E_Y. To prove this, we need only show that $y(q)x_q = x$ on K_q for each q in E_Y, since $g(q)y(q) = x_q(P^* q)y(q)$ and $Px(q) = x(P^* q)$. But for each p in $E(K_q)$ we have $y(q)x_q(p) = y(p)f(p) = x(p)$; that is, $y(q)x_q = x$ on $E(K_q)$. Hence $y(q)x_q = x$ on K_q, thus completing the proof.

COROLLARY 1. $\pi(S_r) = S_r$.

PROOF. We continue to use the notation in the proof of Theorem 1, but now assume that X is a regular square space. We will show that $Y = PX$ is regular by proving that E_Y is completely regular. We will need the fact that both X and Y are G-spaces (since they are square). Let F be structurally closed in E_Y and $q \in E_Y$ with $q \notin F$. Then, by Lemma 2, there is y in Y with $y = 0$ on F and $y(q) \neq 0$. Let $F' = \{p \in E\colon y(p) = 0\}$. Then F' is structurally closed in E, $E(K_r) \subset F'$ for each r in F, and $F' \cap E(K_q) = \varnothing$. Since E is completely regular (Lemma 3) and $E(K_q)$ is compact, we can use a standard compactness argument as in the proof of Theorem 1 to obtain an f in $C_s^b(E)$ for which $f = 0$ on F' and $f \geqslant 1$ on $E(K_q)$. Let g in $C_s^b(E_Y)$ be defined (in terms of f) as in the proof of Theorem 1. Then $g = 0$ on F and $g(q) \geqslant 1$. Thus E_Y is completely regular.

THEOREM 2. $\pi(M_r) = \pi(S_r) = S_r$.

PROOF. In view of Proposition 1 and Corollary 1, it suffices to prove that $S_r \subset \pi(M_r)$. Let $Y \in S_r$. Then, since Y is square, $K = [0,1]E_Y$ is weak* compact and Y may (and will) be identified with the space of all weak* continuous functions f on K satisfying $f(\lambda q) = \lambda f(q)$ for all q in E_Y and $|\lambda| \leqslant 1$ [3, Proof of Theorem 1], [8, Lemma 2]. Let X be the space of all weak* continuous functions f on K satisfying $f(\lambda q) = \lambda f(q)$ for all q in E_Y and $0 \leqslant \lambda \leqslant 1$. (Both X and Y have the uniform norm.) Then X is an M-space, and the map $P\colon X \to X$ defined by $Px(k) = \frac{1}{2}(x(k) - x(-k))$, for x in X and k in K, is a contractive projection of X onto Y. For each q in E_Y let $\delta(q)$ in X^* be the evaluation functional defined on X by $\delta(q)(x) = x(q)$, and let

$E^+ = \{\delta(q): q \in E_Y\}$. Then E is the union of the disjoint sets E^+ and $-E^+$ because the Choquet boundary for X is $E_Y \cup \{0\}$. (See [5, Lemma 4.1].) Let i^* be the adjoint of the inclusion map $i: Y \to X$. Then i^* is a structurally continuous map of E onto E_Y. (One can use Lemma 2 to verify continuity. Also note that $i^* \circ \delta$ is the identity map on E_Y.) To prove E is regular, let $x \in X$ and $p \in V_x$. We will show there is a structurally closed neighborhood W of p such that $W \subset V_x$. (See Lemma 2 for the notation V_x.) We may assume $p \in E^+$ and $x(p) > 0$ since structurally closed (as well as open) sets in E are symmetric. Let $y \in Y$ with $0 < y(p) < x(p)$. Then, since Y is regular, Lemma 3 implies there is g in $C_s^b(E_Y)$ with $g(i^*p) = 1$ and $g = 0$ on $\{q \in E_Y: y(q) = 0\}$. Let $f = g \circ i^*$. Then f is structurally continuous on E, $f(p) = 1$, and $f = 0$ on $\{t \in E: y(t) = 0\}$. Let $z = (x - y) \wedge y$ on K and let $F = \{t \in E: z^-(t) = 0\}$. Then F is closed since $z^- = (-z) \vee 0$ is in X; and F is a neighborhood of p since $p \in V_{z^+} \subset F$. To verify the latter inclusion, let $t \in E$ with $z^+(t) \neq 0$. Then $t = \pm\delta(q)$ for some q in E_Y, and so $z^+(t) = \pm z^+(q)$. Therefore $z^+(q) \neq 0$, hence $z(q) > 0$. Thus $z^-(q) = 0$, and this implies $z^-(t) = 0$. We shall also need the fact that if $t \in F$ and $x(t) = 0$, then $y(t) = 0$. (This is true since if $t = \pm\delta(q)$, then

$$z^-(t) = 0 \Rightarrow z^-(q) = 0 \Rightarrow z(q) \geqslant 0 \Rightarrow (x - y)(q) \geqslant 0$$

and $y(q) \geqslant 0 \Rightarrow x(q) \geqslant y(q) \geqslant 0$.) Let W be the intersection of F and the set $\{t \in E: f(t) \geqslant \frac{1}{2}\}$. Then W is a closed neighborhood of p. Further, $W \subset V_x$. For, suppose not. Then there is t in W with $x(t) = 0$. Then, since $t \in F$, $y(t) = 0$. But this implies $f(t) = 0$, which contradicts $f(t) \geqslant \frac{1}{2}$. Therefore $W \subset V_x$, and this completes the proof that $X \in M_r$.

Remark. In the above proof, if we assume Y is square but not necessarily regular, then X is "almost square" in the sense that if p_1 and p_2 are evaluation functionals at linearly independent points q_1 and q_2 of E_Y and g in $C_s^b(E_Y)$ separates q_1 and q_2, then $g \circ i^*$ in $C_s^b(E)$ will separate p_1 and p_2. Therefore the problem of showing X square reduces to that of separating $\delta(q)$ and $\delta(-q)$ by a function in $C_s^b(E)$ for each q in E_Y. Because the functions in X are positive-homogeneous, the question becomes: Given q in E_Y, is there a bounded weak* continuous function g on $(0, 1]E_Y$ which is constant on $(0, 1]r$ for each r in E_Y and such that $g(q) \neq g(-q)$? An affirmative answer would imply that X is square, hence that $\pi(M \cap S) = S$.

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DEPARTMENT OF MATHEMATICS, ROSEMONT COLLEGE, ROSEMONT, PENNSYLVANIA 19010