CONTRACTIVE PROJECTIONS IN SQUARE BANACH SPACES

NINA M. ROY

ABSTRACT. It is proved that if $X$ is a square space and $P$ is a contractive projection in $X$, then $PX$ is square; and if $X$ is regular, then $PX$ is regular. It is also shown that a regular square space is isometric to the image, under a contractive projection, of a regular (square) Kakutani $M$-space. These results are analogous to those obtained for other classes of $L_1$-preduals by Lindenstrauss and Wulbert, and in this paper their diagram of $L_1$-preduals is enlarged so as to include the classes of square, regular square and regular $M$-spaces.

A square space is a real Banach space which is (linearly) isometric to a uniformly closed linear space $X$ of real functions on a compact Hausdorff space $\Omega$ satisfying:

(i) $|x|$ is upper semicontinuous for each $x$ in $X$, and

(ii) $X$ is invariant under multiplication by functions in $C(\Omega)$.

Square spaces were introduced in [3] and characterized in [8] in terms of the (Alfsen-Effros) structure topology on the set $E$ of extreme points of the dual ball. (See Lemma 1 below.) The class of square spaces, denoted here by $S$, is properly contained in the class $G$ of Grothendieck $G$-spaces [3, Theorems 1, 2] as is the class $M$ of Kakutani $M$-spaces. We call a $G$-space regular if the structure topology on $E$ is regular, and we denote by $M_r$ (respectively, $S_r$) the class of regular $M$-spaces (resp., regular square spaces).

If $B$ is a class of Banach spaces, then, following the notation in [6] and [7], denote by $\pi(B)$ the class of all Banach spaces $Y$ for which there exists an $X$ in $B$ with a subspace $Y'$ isometric to $Y$ and a contractive projection (i.e., a projection of norm 1) of $X$ onto $Y'$. The main results in this paper are $\pi(S) = S$ (Theorem 1) and $\pi(M_r) = \pi(S_r) = S_r$ (Theorem 2).

The diagram below shows how the classes $M_p$, $S_r$, $S$ and $M \cap S$ are related and fit into the diagram of Lindenstrauss spaces ($L_1$-preduals) in [7].

$$
\begin{array}{ccccccc}
C & \rightarrow & C_0 & \rightarrow & M_r & \rightarrow & M \cap S & \rightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_\Sigma & \rightarrow & C_\sigma & \rightarrow & S_r & \rightarrow & S & \rightarrow & G & \rightarrow & \{X: X^* = L_1(\mu)\}
\end{array}
$$

Received by the editors January 9, 1976.


Key words and phrases. Square space, $M$-space, Lindenstrauss space, contractive projection, structure topology, extreme point.

© American Mathematical Society 1976

291
The arrows indicate proper inclusion, and the intersection of two classes is their closest common source, e.g., $M \cap C_\sigma = C_0$. The reader is referred to [6] or [7] for precise definitions of the classes $C$, $C_0$, $C_\Sigma$, $C_\sigma$, $M$ and $G$. (We are writing $C$ in place of $C(K)$, $C_0$ in place of $C_0(K)$, etc.) It is proved in [7] that

$$\pi(C) = \pi(C_0) = \pi(C_\Sigma) = \pi(C_\sigma) = C_\sigma \quad \text{and} \quad \pi(M) = \pi(G) = G.$$ 

The (open) problem of characterizing $\pi(M \cap S)$ is discussed in the Remark at the end of this paper.

After the following preliminaries, we shall verify those parts of our diagram which are not in [7] (or [6]) and then proceed to our main results (Theorems 1 and 2).

In this paper, only real Banach spaces are considered, and we use the following notation and terminology. If $X$ is a Banach space, then $E$ (or, when necessary, $E_X$) denotes the set $\text{ext} B(X^*)$ of extreme points of the closed unit ball $B(X^*)$ in the dual of $X$, and $\overline{E}$ is the weak* closure of $E$. We shall frequently regard $X$ as a space of functions on $E$ (or $E$) and write $x(p)$ in place of $x(p)$ for $x$ in $X$ and $p$ in $E$. We denote by $\mathfrak{M}(X)$ the algebra of multipliers of $X$; by definition, a bounded real function $f$ on $E$ belongs to $\mathfrak{M}(X)$ if for each $x$ in $X$ there is $y$ in $X$ satisfying $f(p)x(p) = y(p)$ for all $p$ in $E$. (This algebra was introduced in [2].) The (Alfsen-Effros) structure topology on $E$ is the topology whose nonempty closed sets are of the form $E \cap N$, where $N$ is a nonzero weak* closed $L$-summand in $X^*$ [1, Proposition 3.3]. Let $C^b_s(E)$ denote the algebra of all bounded, structurally continuous functions on $E$. An important result, used several times in this paper, is that $C^b_s(E) = \mathfrak{M}_s(E)$ [1, Theorem 4.9].

**Lemma 1** [8, Corollary 1]. A Banach space $X$ is square if and only if $C^b_s(E)$ separates linearly independent points in $E$.

**Lemma 2** [8, Lemma 5]. If $X$ is a $G$-space, then a base for the structure topology on $E$ is the family $\{V_x: x \in X\}$, where $V_x = \{p \in E: x(p) \neq 0\}$.

**Lemma 3.** If $X$ is a $G$-space for which $E$ is (structurally) regular, then $E$ is completely regular.

**Proof.** Let $E_\sigma = \{(p,-p): p \in E\}$ have the quotient structure topology. Then by [8, Theorem 2] and the fact that the closed sets in $E$ are symmetric, we have the implications

$$E \text{ regular } \Rightarrow E_\sigma \text{ regular } \Rightarrow E_\sigma \text{ completely regular } \Rightarrow E \text{ completely regular.}$$

The following three propositions are useful in verifying the diagram.

**Proposition 1.** $M_\tau = M \cap S_\tau$.

**Proof.** It suffices to show that every regular $G$-space is square. Let $X$ be a regular $G$-space. Then $E$ is completely regular by Lemma 3. Since $\{p,-p\}$ is
closed for each $p$ in $E$ (because $X$ is a Lindenstrauss space) it follows from Lemma 1 that $X$ is square.

The next two results follow immediately from the proofs of [8, Corollaries 2, 4] and the fact that if $E_\alpha$ is completely regular, then so is $E$.

**Proposition 2.** Every separable $G$-space is in $S_r$.

**Proposition 3.** $C_\alpha \subseteq S_r$.

In view of the above propositions as well as results in [3] and [7], it is clear that to complete the verification of our diagram we need only show that those inclusions not appearing in [7] are all proper. Cunningham established [3] that $M \cap S \neq M$, $M \cap S \neq S$, and $S \neq G$; the example in [3, Theorem 3] together with Proposition 3 show that $M_r \neq S_r$. Effros constructed a separable $M$-space which is not a $C_\alpha$-space [5, Theorem 4.3]; hence $C_\alpha \neq S_r$ by Proposition 2 (also $C_0 \neq M_r$). Finally, a square space $X$ which is not regular can be found in [8, pp. 147, 148]; that $X$ is also an $M$-space can be verified as follows. By definition, $X$ is the uniform closure of a linear space $Y$ of functions (of the form $f + gx_1$, in the notation of [8]) on a compact ordinal space $\Omega$. One first shows that if $y \in Y$, then for each $t$ in $\Omega$ there is $y_t$ in $Y$ such that $|y(t)| = y_t(t)$ and $|y| - y_t$ is continuous at $t$. (Consider the three cases $t \in F$, $t \in L - F$, $t \notin L$.) Then the partition of unity technique used in [3, Proof of Theorem 1, p. 555] can be applied to show that if $y \in Y$, then $|y| \in X$. Therefore if $x \in X$, then $x^+ = (x + |x|)/2$ is in $X$, and so $X$ is a closed linear sublattice of $(m(\Omega), \text{hence an } M$-space. This example proves that $M_r \neq M \cap S$ and $S_r \neq S$, hence the diagram is verified.

In what follows, $E = \text{ext } B(X^*)$, $E_Y = \text{ext } B(Y^*)$, and $E(K_\alpha) = \text{ext } K_\alpha$.

**Theorem 1.** $\pi(S) = S$.

**Proof.** Let $X$ be a square space and $P$ a contractive projection on $X$. We shall use Lemma 1 to show that $Y = PX$ is square.

For each $q$ in $E_Y$, let $K_q$ denote the set of all norm-preserving extensions of $q$ in $X^*$. Then $K_q$ is a convex weak$^*$ compact subset of $B(X^*)$, $P^*q \in K_q$, and $E(K_q) \subseteq E$. ($P^*$ denotes the adjoint of $P$.) Further, $E(K_q)$ is weak$^*$ compact since $X$ is square. For if $\{p_\alpha\}$ is a net in $E(K_q)$, then there is a subnet $\{p_\beta\}$ which converges weak$^*$ to some $p$ in $K_q$. Then $p$ is in $E \cap K_q$, hence in $E \cap K_q$ because $||p|| = 1$ and $E \subseteq [0, 1]E$ [3, Lemma 2], [8, Lemma 2]. Thus $p \in E(K_q)$. It now follows that $E(K_q)$ is structurally compact since the structure topology on $E$ is smaller than the (relativized) weak$^*$ topology on $E$.

Let $q_1$ and $q_2$ be linearly independent points in $E_Y$ and let $a$ and $b$ be real numbers with $0 < a < b < 1$. Then there is $f$ in $C^b_s(E)$ such that $f < a$ on $E(K_{q_1})$ and $f > b$ on $E(K_{q_2})$. To prove this, we use the "only if" part of Lemma 1 and the structural compactness of the $E(K_{q_i})$. If $s \in E(K_{q_i})$, then for each $p$ in $E(K_{q_i})$ there is $f_p$ in $C^b_s(E)$ with $f_p(p) = 0$ and $f_p(s) = 1$; let $U_p = \{r \in E(K_{q_i}) : f_p(r) < a\}$. Then a finite number of these open sets, say $U_{p_1}, U_{p_2}, \ldots, U_{p_n}$, cover $E(K_{q_i})$. Let $f_s = f_{p_1} \wedge f_{p_2} \wedge \cdots \wedge f_{p_n}$. Then $f_s$
For each $s$ in $E(K_q)$ choose such a function $f_s$ and define $V_s = \{ t \in E(K_q) : f_s(t) > b \}$. Then $E(K_q)$ is covered by a finite number of these open sets, say $V_{q_1}, V_{q_2}, \ldots, V_{q_m}$. Let $f = f_{q_1} \vee f_{q_2} \vee \cdots \vee f_{q_m}$. Then $f \in C^b_s(E), f < a$ on $E(K_q)$ and $f > b$ on $E(K_q)$.

Let $q_1, q_2, a, b$ and $f$ be as above. We will show there is $g$ in $C^b_s(S_r)$ with $g(q_1) < a$ and $g(q_2) > b$, thus proving $Y$ is square. For each $q$ in $E_Y$ we have that $f$ is weak* continuous on the compact set $E(K_q)$; hence there is $x_q$ in $X$ such that $f = x_q$ on $E(K_q)$ and $\|x_q\| = \|f|_{E(K_q)}\|$ [4, Lemma 6.1]. Define $g$ on $E_y$ by $g(q) = x_q(P^* q)$ for each $q$ in $E_y$. For each $p$ in $E(K_q)$ we have $x_q(p) = f(p) < a$; hence $x_q \leq a$ on $K_q = \overline{co} E(K_q)$, and so $g(q_1) = x_{q_1}(P^* q_1) < a$. Similarly $g(q_2) > b$. We now show $g$ is in $C^b_s(E_Y)$. Clearly $g$ is bounded, indeed $\|g(q)\| \leq \|f\|$ for each $q$ in $E_y$. Let $y \in Y$. Then, since $f \in \mathcal{F}(X)$, there is $x$ in $X$ such that $f = x$ on $E_Y$. It follows that $gy = Px$ on $E_y$. To prove this, we need only show that $y(q)x_q = x$ on $K_q$ for each $q$ in $E_y$, since $g(q)y(q) = x_q(P^* q)y(q)$ and $Px(q) = x(P^* q)$. But for each $p$ in $E(K_q)$ we have $y(q)x_q(p) = y(p)f(p) = x(p)$; that is, $y(q)x_q = x$ on $E(K_q)$. Hence $y(q)x_q = x$ on $K_q$, thus completing the proof.

Corollary 1. $\pi(S_r) = S_r$.

Proof. We continue to use the notation in the proof of Theorem 1, but now assume that $X$ is a regular square space. We will show that $Y = PX$ is regular by proving that $E_Y$ is completely regular. We will need the fact that both $X$ and $Y$ are $G$-spaces (since they are square). Let $F$ be structurally closed in $E_Y$ and $q \in E_Y$ with $q \not\in F$. Then, by Lemma 2, there is $y$ in $Y$ with $y = 0$ on $F$ and $y(q) \neq 0$. Let $F^* = \{ p \in E : y(p) = 0 \}$. Then $F^*$ is structurally closed in $E, E(K_q) \subset F^*$ for each $r$ in $F$, and $F^* \cap E(K_q) = \emptyset$. Since $E$ is completely regular (Lemma 3) and $E(K_q)$ is compact, we can use a standard compactness argument as in the proof of Theorem 1 to obtain an $f$ in $C^b_s(E)$ for which $f = 0$ on $F^*$ and $f \geq 1$ on $E(K_q)$. Let $g$ in $C^b_s(E_Y)$ be defined (in terms of $f$) as in the proof of Theorem 1. Then $g = 0$ on $F$ and $g(q) \geq 1$. Thus $E_Y$ is completely regular.

Theorem 2. $\pi(M_r) = \pi(S_r) = S_r$.

Proof. In view of Proposition 1 and Corollary 1, it suffices to prove that $S_r \subset \pi(M_r)$. Let $Y \in S_r$. Then, since $Y$ is square, $K = [0, 1]E_Y$ is weak* compact and $Y$ may (and will) be identified with the space of all weak* continuous functions $f$ on $K$ satisfying $f(\lambda q) = \lambda f(q)$ for all $q$ in $E_Y$ and $|\lambda| \leq 1$ [3, Proof of Theorem 1], [8, Lemma 2]. Let $X$ be the space of all weak* continuous functions $f$ on $K$ satisfying $f(\lambda q) = \lambda f(q)$ for all $q$ in $E_Y$ and $0 \leq \lambda \leq 1$. (Both $X$ and $Y$ have the uniform norm.) Then $X$ is an $M$-space, and the map $P : X \to X$ defined by $Px(k) = \frac{1}{2}(x(k) - x(-k))$, for $x$ in $X$ and $k$ in $K$, is a contractive projection of $X$ onto $Y$. For each $q$ in $E_Y$ let $\delta(q)$ in $X^*$ be the evaluation functional defined on $X$ by $\delta(q)(x) = x(q)$, and let
$E^+ = \{\delta(q): q \in E_Y\}$. Then $E$ is the union of the disjoint sets $E^+$ and $-E^+$ because the Choquet boundary for $X$ is $E_Y \cup \{0\}$. (See [5, Lemma 4.1].) Let $i^*$ be the adjoint of the inclusion map $i: Y \to X$. Then $i^*$ is a structurally continuous map of $E$ onto $E_Y$. (One can use Lemma 2 to verify continuity. Also note that $i^* \circ \delta$ is the identity map on $E_Y$.) To prove $E$ is regular, let $x \in X$ and $p \in V_x$. We will show there is a structurally closed neighborhood $W$ of $p$ such that $W \subset V_x$. (See Lemma 2 for the notation $V_x$.) We may assume $p \in E^+$ and $x(p) > 0$ since structurally closed (as well as open) sets in $E$ are symmetric. Let $y \in Y$ with $0 < y(p) < x(p)$. Then, since $Y$ is regular, Lemma 3 implies there is $g$ in $C^b_y(E_Y)$ with $g(i^*p) = 1$ and $g = 0$ on $\{q \in E_Y: y(q) = 0\}$. Let $f = g \circ i^*$. Then $f$ is structurally continuous on $E$, $f(p) = 1$, and $f = 0$ on $\{t \in E: y(t) = 0\}$. Let $z = (x - y) \wedge y$ on $K$ and let $F = \{t \in E: z^-(t) = 0\}$. Then $F$ is closed since $z^- = (-z) \vee 0$ is in $X$; and $F$ is a neighborhood of $p$ since $p \in V_x \subset F$. To verify the latter inclusion, let $t \in E$ with $z^+(t) \neq 0$. Then $t = \pm \delta(q)$ for some $q \in E_Y$, and so $z^+(t) = \pm z^+(q)$. Therefore $z^+(q) \neq 0$, hence $z(q) > 0$. Thus $z^-(q) = 0$, and this implies $z^-(t) = 0$. We shall also need the fact that if $t \in F$ and $x(t) = 0$, then $y(t) = 0$. (This is true since if $t = \pm \delta(q)$, then

$$z^-(t) = 0 \Rightarrow z^-(q) = 0 \Rightarrow x(q) > 0 \Rightarrow (x - y)(q) > 0$$

and $y(q) > 0 \Rightarrow x(q) > y(q) > 0$.) Let $W$ be the intersection of $F$ and the set $\{t \in E: f(t) > \frac{1}{2}\}$. Then $W$ is a closed neighborhood of $p$. Further, $W \subset V_x$. For, suppose not. Then there is $t \in W$ with $x(t) = 0$. Then, since $t \in F$, $y(t) = 0$. But this implies $f(t) = 0$, which contradicts $f(t) > \frac{1}{2}$. Therefore $W \subset V_x$, and this completes the proof that $X \subset M_y$.

REMARK. In the above proof, if we assume $Y$ is square but not necessarily regular, then $X$ is "almost square" in the sense that if $p_1$ and $p_2$ are evaluation functionals at linearly independent points $q_1$ and $q_2$ of $E_Y$ and $g$ in $C^b_y(E_Y)$ separates $q_1$ and $q_2$, then $g \circ i^*$ in $C^b_y(E)$ will separate $p_1$ and $p_2$. Therefore the problem of showing $X$ square reduces to that of separating $\delta(q)$ and $\delta(-q)$ by a function in $C^b_y(E)$ for each $q$ in $E_Y$. Because the functions in $X$ are positive-homogeneous, the question becomes: Given $q$ in $E_Y$, is there a bounded weak* continuous function $g$ on $(0, 1]\E_Y$ which is constant on $(0, 1)\E$ for each $r$ in $E_Y$ and such that $g(q) \neq g(-q)$? An affirmative answer would imply that $X$ is square, hence that $\pi(M \cap S) = S$.

ACKNOWLEDGEMENT. I would like to thank F. Cunningham, Jr. for suggesting the possibility $S \subset \pi(M \cap S)$, which led to Theorem 2 of this paper.

REFERENCES


Department of Mathematics, Rosemont College, Rosemont, Pennsylvania 19010