PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, September 1976

THE ORBIT SPACE OF A SPHERE BY AN ACTION OF Z_{p^s}

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ABSTRACT. Let X be a finite CW complex with the Z_{p^r} homology of an n-sphere. Suppose Z_{p^s} acts cellularly on X. The homology of the orbit space X/Z_{p^s} with coefficients Z_{p^r} is computed.

Introduction. Let X be a finite CW complex. Denote by Z_m the cyclic group of order m. If $n|m$, then Z_n is naturally identified with a subgroup of Z_m. The group Z_1 is the identity group. A cellular *action* of Z_m on X is a cellular map $\alpha: X \to X$ such that α^m equals the identity map. If H is a subset of Z_m, H may be identified with a collection of maps α^i, and the set of points in X left fixed by each element of H is denoted X^H. If we identify a point $x \in X$ with $\alpha(x)$, we obtain the orbit space X/Z_m. If $Z_n \subset Z_m$, then X^{Z_n} inherits a $Z_{m/n}$ action, and $X^{Z_n}/Z_{m/n}$ is naturally contained in X/Z_m. The (-1)-sphere is, by definition, the empty set.

In this paper we shall assume p is a prime, X has the Z_{p^r} homology of an n-sphere, and Z_{p^s} acts cellularly on X. We shall then compute the homology of the orbit space X/Z_{p^s}. In particular, we prove the following theorem.

THEOREM A. *Let p be an odd prime integer, and let $r \geqslant s$. Suppose X is a finite CW complex with the Z_{p^r} homology of an n-sphere, and Z_{p^s} acts cellularly on X. Assume, for $l = 0, 1, \ldots, s$, that $X^{Z_{p^{(s-l)}}}$ has the Z_{p^r} homology of a k_l-sphere (so $k_0 \leqslant k_1 \leqslant \cdots \leqslant k_s = n$). Then $H_i(X/Z_{p^s}; Z_{p^r})$ equals Z_{p^r} for $i = 0$; 0 for $1 \leqslant i < k_0 + 2$; Z_p for $k_0 + 2 \leqslant i < k_1 + 2; \ldots; Z_{p^j}$ for $k_{j-1} + 2 \leqslant i < k_j + 2; \ldots; Z_{p^s}$ for $k_{s-1} + 2 \leqslant i < k_s = n$; Z_{p^r} for $i = n$; 0 for $i > n$.*

The restriction that p be odd is for convenience. In fact, one needs only that for each i either $k_i = k_{i+1}$ or $k_i \leqslant k_{i+1} - 2$; this property is well known if p is odd. If $p = 2$ and for some j, $k_j = k_{j+1} - 1$, the formulas in Theorem A need modification; in this case the change of groups is delayed by one, so that $H_{k_j+2}(X/Z_{p^s}; Z_{p^r})$ is set equal to the group (already computed) $H_{k_j}(X/Z_{p^s}; Z_{p^r})$. Thus, if $s = 5$, $k_0 = 1$, $k_1 = 3$, $k_2 = 4$, $k_3 = 5$, $k_4 = 7$, $k_5 = 9$, we obtain that $H_i(X/Z_{p^s}; Z_{p^r})$ equals 0 for $1 \leqslant i \leqslant 2$; Z_p for $3 \leqslant i \leqslant 6$; Z_{p^4} for $7 \leqslant i \leqslant 8$; Z_{p^r} for $i = 9$.

The assumption that $X^{Z_{p^i}}$ has the Z_{p^r} homology of a sphere is no restriction at all; this is an easy application of the Smith theorem and the Universal Coefficient Theorem. (It is not hard to see that if Y has the Z_{p^r} homology of

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Presented to the Society, August 19, 1975 under the title *Applications of an equivariant universal coefficient theorem*; received by the editors February 28, 1975.

AMS (MOS) subject classifications (1970). Primary 55C35; Secondary 57E25.

an n-sphere, then $H_n(Y; Z)$ contains a free summand Z; and $H_i(Y; Z)$ contains no p-torsion for any i.)

We note that if one desires $H_i(X/Z_{p^s}; Z_{p^j})$ where $1 \leqslant j < s$, by an easy application of the Universal Coefficient Theorem, one need only tensor the group obtained in Theorem A with Z_{p^j}; one uses the fact that $Z_{p^j} \otimes Z_{p^k} = Z_{p^m}$ where m is the minimum of j and k.

Moreover, if q is a prime other than p and $R = Z_{q^r}$ for some r or R is the field of rational numbers, then $H_i(X/Z_{p^s}; R) = H_i(X; R)^{Z_{p^s}}$, the subgroup of $H_i(X; R)$ left fixed by the homology map induced by α. (See, for example, [1, p. 37].) Hence our result completely determines the integral homology groups of X/Z_{p^s}.

The proof of Theorem A is based on our paper [4]. We shall briefly summarize here the salient features of that paper: Suppose R is a commutative ring. One may construct from R and the group Z_{p^s} a ring $\mathcal{I}$, called the isotopy ring. Suppose $h_i(X, A)$ is an equivariant homology theory defined for pairs of finite CW complexes with cellular Z_{p^s} actions; assume $h_i(Z_{p^s}/K) = 0$ for all $i > 0$ and all subgroups K of Z_{p^s}; and that $h_0(Z_{p^s}/K)$ is an R module for each K. Then one may construct a left $\mathcal{I}$ module M with the following property: For any pair (X, A) of finite CW complexes with cellular Z_{p^s} action, there is a first quadrant spectral sequence with $E^2_{m,n} = \operatorname{Tor}^{\mathcal{I}}_m({}_G H_n(X, A; \mathcal{I}), M)$ which converges to $h_*(X, A)$. Here ${}_G H_n(X, A; \mathcal{I})$ is a particular right $\mathcal{I}$ module with the property that, as an R module, ${}_G H_n(X, A; \mathcal{I}) = \oplus H_n(X^K, A^K; R)$, where the summation runs over all subgroups K of Z_{p^s}.

An example. In the proof of Theorem A, it will be convenient to have an example of a Z_{p^s} action on a Z_{p^r}-n-sphere. Ultimately, the example will save us some messy algebra in the computation of the E^2 terms of various spectral sequences.

Let ρ_j denote the complex numbers with the (linear) action $g \cdot v = \exp((2\pi i)/p^{(s-j)})v$, where g generates Z_{p^s}. Then

$$m_0\rho_0 \oplus m_1\rho_1 \oplus \cdots \oplus m_s\rho_s \quad (\text{for } m_i \geqslant 0)$$

is a vector space of dimension $2(m_0 + m_1 + \cdots + m_s)$ over the reals. Let X denote its unit sphere, so that X is an $n = 2(m_0 + \cdots + m_s) - 1$-sphere. Let $k_l = 2(m_{s-l} + m_{s-l+1} + \cdots + m_s) - 1$ for $0 \leqslant l \leqslant s$. Then X has a Z_{p^s} action, and $X^{Z_{p^{(s-l)}}}$ is a k_l-sphere.

Steenrod and Epstein [3, p. 67] show how to obtain a convenient cell decomposition of X so that g becomes a cellular map. If, for example, the unit sphere S of $\rho_{s-2} \oplus m_{s-1}\rho_{s-1} \oplus m_s\rho_s$ has been given a cell decomposition already and $m_{s-2} > 1$, then we obtain a cell decomposition of the unit sphere T of $2\rho_{s-2} \oplus m_{s-1}\rho_{s-1} \oplus m_s\rho_s$ as follows: The sphere of ρ_{s-2} has a cell decomposition with p^2 0-cells $e^0, ge^0, \ldots, g^{p^2-1}e^0$ and p^2 1-cells $e^1, ge^1, \ldots, g^{p^2-1}e^1$. For the i-cells of T, $i \leqslant k = 2(m_{s-1} + m_s + 1) - 1$, the dimension of S, we use the cells of S. T has p^2 $(k+1)$-cells, namely $S * g^ie^0$ (the join); and p^2 $(k+2)$-cells, namely $S * g^ie^1$.

In this manner we obtain a cell decomposition for X with 1 i-cell e^i if $0 \leqslant i \leqslant k_0$; with p i-cells $e^i, ge^i, \ldots, g^{p-1}e^i$ if $k_0 + 1 \leqslant i \leqslant k_1$; $\ldots$; with p^m i-cells $e^i, ge^i, \ldots, g^{p^m-1}e^i$ if $k_{m-1} + 1 \leqslant i \leqslant k_m$. It is easy to see that

$g(g^je^i) = g^{j+1}e^i$ where $j+1$ is reduced modulo the relevant power of p. Moreover if i is even and $k_{m-1}+1 < i \leqslant k_m$, then

$$\partial(g^je^i) = \sum_{l=0,p^m-1} g^l e^{i-1}.$$

If $i = k_{m-1}+1$,

$$\partial(g^je^i) = \sum_{l=0,\dots,p^{m-1}-1} g^l e^{i-1}.$$

If i is odd and $k_{m-1}+1 < i \leqslant k_m$, then

$$\partial(g^je^i) = g^{j+1}e^{i-1} - g^je^{i-1}.$$

Note that by suspending the above X, we may ensure that k_0 be even if desired. We obtain readily the following facts about this X.

LEMMA 1. *Let $r \geqslant s$. Suppose $0 \leqslant k_{s-2} < k_{s-1} < k_s$. Then*
$H_i(X/Z_{p^s}, X^{Z_p}/Z_{p^{s-1}}; Z_{p^r})$ equals 0 for $i \leqslant k_{s-1}$; Z_{p^r} for $i = k_{s-1}+1$; Z_{p^s} for $k_{s-1}+2 \leqslant i < k_s$.
$H_i(X/Z_{p^s}, X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r})$ equals 0 for $i \leqslant k_{s-2}$; Z_{p^r} for $i = k_{s-2}+1$; $Z_{p^{s-1}}$ for $k_{s-2}+2 \leqslant i < k_{s-1}$.

PROOF. A simple exercise. Q.E.D.

Proofs.

LEMMA 2. *Let $r \geqslant s$. Suppose X is a finite CW complex with the Z_{p^r} homology of an n-sphere. Let Z_{p^s} act cellularly on X, so that X^{Z_p} has the Z_{p^r} homology of a k-sphere, $0 \leqslant k < n$. Then $H_i(X/Z_{p^s}, X^{Z_p}/Z_{p^{s-1}}; Z_{p^r})$ equals 0 for $0 \leqslant i \leqslant k$; Z_{p^r} for $i = k+1$; Z_{p^s} for $k+2 \leqslant i < n$; Z_{p^r} for $i = n$; 0 for $i > n$.*

PROOF. Let Γ be the left $\mathcal{G}$ module corresponding to the homology theory

$$h_i(X,A) = H_i(X/Z_{p^s}, X^{Z_p}/Z_{p^{s-1}} \cup A/Z_{p^s}; Z_{p^r}).$$

There is a spectral sequence with $E^2_{a,b} = \operatorname{Tor}^{\mathcal{G}}_a({}_GH_b(X;\mathcal{G}),\Gamma)$ converging to $h_i(X)$. Let k_l be the dimension of $X^{Z_{p^{(s-l)}}}$. We assume first that $k_0 > 0$. Note that $E^2_{a,b} = 0$ for $0 < b < k_0$. Hence, for $b < k_0$, $E^2_{b,0} = h_b(X)$ for any X with the assumed properties. Using Lemma 1, and noting that $E^2_{b,0}$ is independent of k_0 (as long as $0 < k_0$), we see $E^2_{b,0} = 0$ for all b.

Now, since $E^2_{a,b} = 0$ for $k_0 < b < k_1$, it follows $E^2_{a,k_0} = h_{a+k_0}(X)$ for any such X. Using our example, $E^2_{a,k_0} = 0$ for all a. Continuing in this manner, we see $E^2_{a,b} = 0$ for $a < k_{s-1} = k$. But $E^2_{a,b} = 0$ for $k < b < k_s = n$. Hence $E^2_{a,k} = h_{a+k}(X)$ for any such X. By Lemma 1, using the independence of $H_k(X;\mathcal{G})$ from n, we see $E^2_{0,k} = 0$; $E^2_{1,k} = Z_{p^r}$; $E^2_{i,k} = Z_{p^s}$ for $i \geqslant 2$. Thus we obtain the lemma for $i < n$. It is well known that $h_i(X) = 0$ for $i > n$. (See, for example, [1, p. 43].) Finally, $E^2_{n,0} = {}_GH_n(X;\mathcal{G}) \otimes \Gamma = Z_{p^r}$, and d: $E^2_{n-k+1,k} \to E^2_{n,0}$ becomes $d\colon Z_{p^s} \to Z_{p^r}$. If d were not one-to-one, then $h_{n+1}(X)$ would not equal zero. Hence $E^\infty_{n,0} = Z_{p^{r-s}}$, $E^\infty_{n-k,k} = Z_{p^s}$, and the order of $h_n(X)$ is p^r. The case $r = 1$ would show that $h_n(X) = Z_p$. From this fact, a consideration of cases and the Universal Coefficient Theorem, using the fact that $h_{n+1}(X) = 0$, shows $h_n(X) = Z_{p^r}$.

Minor modifications in the above argument yield the result if $k_0 = k_1 = \cdots = k_j = -1$ for some $j < s - 1$. Q.E.D.

LEMMA 3. *Let $r \geqslant s$. Let X be as in the statement of Theorem* A. *Suppose $0 \leqslant k_{s-2} < k_{s-1} < n$. Let $k = k_{s-2}$. Then $H_i(X/Z_{p^s}, X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r})$ equals* 0 *for* $0 \leqslant i \leqslant k$; Z_{p^r} *for* $i = k + 1$; $Z_{p^{s-1}}$ *for* $k + 2 \leqslant i \leqslant k_{s-1}$.

PROOF. The proof is completely analogous to that of Lemma 2, using $h_i(X,A) = H_i(X/Z_{p^s}, X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r})$, a corresponding left $\mathcal{G}$ module Γ, and the fact that

$$E^2_{0,k_{s-1}} = {}_G H_{k_{s-1}}(X; \mathcal{G}) \otimes_{\mathcal{G}} \Gamma = 0.$$

Q.E.D.

Proof of Theorem A. We prove Theorem A by induction on s. If $s = 1$, we let Θ be the left $\mathcal{G}$ module corresponding to $h_i(X,A) = H_i(X/Z_p, A/Z_p; Z_{p^r})$. The spectral sequence converging to $h_i(X)$ has $E^2_{a,b} = 0$ for $b \neq 0, k_0, k_1$. Assuming $0 < k_0 < k_1 = n$ we see $E^2_{a,0} = h_a(X)$ for any such X, if $0 \leqslant a < k_0$. Hence by use of our example, $E^2_{0,0} = Z_{p^r}$; $E^2_{a,0} = 0$ for $a > 0$. Hence $E^2_{a,k_0} = h_{a+k_0}(X)$ for any such X, if $a + k_0 < k_1$. By use of our example, $E^2_{0,k_0} = E^2_{1,k_0} = 0$; $E^2_{a,k_0} = Z_p$ for $a \geqslant 2$. Finally,

$$E^2_{0,k_1} = {}_G H_{k_1}(X; \mathcal{G}) \otimes \Theta = Z_{p^r}.$$

We obtain our result immediately for $0 \leqslant i < n$ and $i > n$; for the case $i = n$ we argue as at the end of Lemma 2. The case $k_0 \leqslant 0$ is handled similarly.

We now assume Theorem A for $s - 1$ and prove it for s. Hence $H_i(X^{Z_p}/Z_{p^{s-1}}; Z_{p^r})$ is known by induction. In particular,

$$H_i(X^{Z_p}/Z_{p^{s-1}}; Z_{p^r}) = 0 \quad \text{for } i > k_{s-1}.$$

Yet $H_i(X/Z_{p^s}, X^{Z_p}/Z_{p^{s-1}}; Z_{p^r}) = 0$ for $0 \leqslant i \leqslant k_{s-1}$ by Lemma 2. The long exact sequence for the pair $(X/Z_{p^s}, X^{Z_p}/Z_{p^{s-1}})$ then implies

$$H_i(X/Z_{p^s}; Z_{p^r}) = H_i(X/Z_{p^s}, X^{Z_p}/Z_{p^{s-1}}; Z_{p^r}) \quad \text{for } i \geqslant k_{s-1} + 2$$

and

$$H_i(X/Z_{p^s}; Z_{p^r}) = H_i(X^{Z_p}/Z_{p^{s-1}}; Z_{p^r}) \quad \text{for } 0 \leqslant i \leqslant k_{s-1} - 1.$$

This yields our result for all i except $i = k_{s-1}$ and $i = k_{s-1} + 1$.

The same long exact sequence implies that

$$0 = H_{k_{s-1}+1}(X^{Z_p}/Z_{p^{s-1}}; Z_{p^r}) \to H_{k_{s-1}+1}(X/Z_{p^s}; Z_{p^r})$$
$$\to Z_{p^r} \to Z_{p^r} \to H_{k_{s-1}}(X/Z_{p^s}; Z_{p^r}) \to 0$$

is exact. Hence $H_{k_{s-1}+1}(X/Z_{p^s}; Z_{p^r})$ and $H_{k_{s-1}}(X/Z_{p^s}; Z_{p^r})$ are both cyclic of the same order.

Assume that $k_{s-2} < k_{s-1}$. Since p is odd, by [2, p. 129] $k_{s-2} \leqslant k_{s-1} - 2$. Hence the long exact sequence for the pair $(X/Z_{p^s}, X^{Z_{p^2}}/Z_{p^{s-2}})$ yields an exact sequence

$$0 = H_{k_{s-1}}(X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r}) \to H_{k_{s-1}}(X/Z_{p^s}; Z_{p^r})$$
$$\to H_{k_{s-1}}(X/Z_{p^s}, X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r}) \to H_{k_{s-1}-1}(X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r}) = 0.$$

Hence, by Lemma 3,

$$H_{k_{s-1}}(X/Z_{p^s}; Z_{p^r}) = Z_{p^{s-1}}$$

and our theorem follows.

If $k_{s-2} = k_{s-1}$ but $k_{s-3} < k_{s-1}$, one deals similarly with the exact sequence of the pair X/Z_{p^s}, $X^{Z_{p^3}}/Z_{p^{s-3}}$. The general result is clear.

If $p = 2$ and $k_{s-2} = k_{s-1} - 1$, then we obtain the exact sequence

$$0 \to H_{k_{s-1}}(X/Z_{p^s}; Z_{p^r}) \to Z_{p^r} \to H_{k_{s-2}}(X^{Z_{p^2}}/Z_{p^{s-2}}; Z_{p^r})$$
$$= Z_{p^r} \to H_{k_{s-2}}(X/Z_{p^s}; Z_{p^r}) = Z_{p^{s-2}} \to 0.$$

Hence, in this case,

$$H_{k_{s-1}}(X/Z_{p^s}; Z_{p^r}) = Z_{p^{s-2}} = H_{k_{s-2}}(X/Z_{p^s}; Z_{p^r}).$$

Q.E.D.

References

1. A. Borel et al., *Seminar on transformation groups*, Ann. of Math. Studies, no. 46, Princeton Univ. Press, Princeton, N.J., 1960. MR **22** #7129.

2. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

3. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies, no. 50, Princeton Univ. Press, Princeton, N.J., 1962. MR **26** #3056.

4. S. J. Willson, *Equivariant homology theories on G-complexes*, Trans. Amer. Math. Soc. 212 (1975), 155–171.

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