THE ORBIT SPACE OF A SPHERE BY AN ACTION OF $\mathbb{Z}_p$.

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Abstract. Let $X$ be a finite CW complex with the $\mathbb{Z}_p$ homology of an $n$-sphere. Suppose $\mathbb{Z}_p$ acts cellularly on $X$. The homology of the orbit space $X/\mathbb{Z}_p$, with coefficients $\mathbb{Z}_p$, is computed.

Introduction. Let $X$ be a finite CW complex. Denote by $\mathbb{Z}_m$ the cyclic group of order $m$. If $n|m$, then $\mathbb{Z}_n$ is naturally identified with a subgroup of $\mathbb{Z}_m$. The group $\mathbb{Z}_1$ is the identity group. A cellular action of $\mathbb{Z}_m$ on $X$ is a cellular map $\alpha: X \to X$ such that $\alpha^m$ equals the identity map. If $H$ is a subset of $\mathbb{Z}_m$, $H$ may be identified with a collection of maps $\alpha^h$, and the set of points in $X$ left fixed by each element of $H$ is denoted $X^H$. If we identify a point $x \in X$ with $\alpha(x)$, we obtain the orbit space $X/\mathbb{Z}_m$. If $\mathbb{Z}_n \subset \mathbb{Z}_m$, then $X^{\mathbb{Z}_n}$ inherits a $\mathbb{Z}_m$ action, and $X^{\mathbb{Z}_n}/\mathbb{Z}_m/n$ is naturally contained in $X/\mathbb{Z}_m$. The $(-1)$-sphere is, by definition, the empty set.

In this paper we shall assume $p$ is a prime, $X$ has the $\mathbb{Z}_p$ homology of an $n$-sphere, and $\mathbb{Z}_p$ acts cellularly on $X$. We shall then compute the homology of the orbit space $X/\mathbb{Z}_p$. In particular, we prove the following theorem.

Theorem A. Let $p$ be an odd prime integer, and let $r \geq s$. Suppose $X$ is a finite CW complex with the $\mathbb{Z}_p$ homology of an $n$-sphere, and $\mathbb{Z}_p$ acts cellularly on $X$. Assume, for $l = 0, 1, \ldots, s$, that $X^{\mathbb{Z}_p^{l+1}}$ has the $\mathbb{Z}_p$ homology of a $k_l$-sphere (so $k_0 \leq k_1 \leq \cdots \leq k_s = n$). Then $H_i(X/\mathbb{Z}_p; \mathbb{Z}_p)$ equals $\mathbb{Z}_p$ for $i = 0$; $0$ for $1 \leq i \leq k_0 + 2$; $\mathbb{Z}_p$ for $k_0 + 2 \leq i < k_1 + 2$; $\cdots$; $\mathbb{Z}_p$ for $k_{j-1} + 2 \leq i < k_j + 2$; $\cdots$; $\mathbb{Z}_p$ for $k_{s-1} + 2 \leq i < k_s = n$; $\mathbb{Z}_p$ for $i = n$; $0$ for $i > n$.

The restriction that $p$ be odd is for convenience. In fact, one needs only that for each $i$ either $k_i = k_{i+1} + 1$ or $k_i < k_{i+1} - 2$; this property is well known if $p$ is odd. If $p = 2$ and for some $j$, $k_j = k_{j+1} - 1$, the formulas in Theorem A need modification; in this case the change of groups is delayed by one, so that $H_{k_j+2}(X/\mathbb{Z}_p; \mathbb{Z}_p)$ is set equal to the group (already computed) $H_{k_j}(X/\mathbb{Z}_p; \mathbb{Z}_p)$. Thus, if $s = 5$, $k_0 = 1$, $k_1 = 3$, $k_2 = 4$, $k_3 = 5$, $k_4 = 7$, $k_5 = 9$, we obtain that $H_i(X/\mathbb{Z}_p; \mathbb{Z}_p)$ equals $0$ for $1 \leq i \leq 2$; $\mathbb{Z}_p$ for $3 \leq i \leq 6$; $\mathbb{Z}_p$ for $7 \leq i \leq 8$; $\mathbb{Z}_p$ for $i = 9$.

The assumption that $X^{\mathbb{Z}_p}$ has the $\mathbb{Z}_p$ homology of a sphere is no restriction at all; this is an easy application of the Smith theorem and the Universal Coefficient Theorem. (It is not hard to see that if $Y$ has the $\mathbb{Z}_p$ homology of...
an $n$-sphere, then $H_n(Y; Z)$ contains a free summand $Z$; and $H_j(Y; Z)$ contains no $p$-torsion for any $i$.

We note that if one desires $H_t(X/Z_p; Z_j)$ where $1 \leq j < s$, by an easy application of the Universal Coefficient Theorem, one need only tensor the group obtained in Theorem A with $Z_p^j$; one uses the fact that $Z_p^j \otimes Z_p^k = Z_p^m$ where $m$ is the minimum of $j$ and $k$.

Moreover, if $q$ is a prime other than $p$ and $R = Z_q^r$ for some $r$ or $R$ is the field of rational numbers, then $H_j(X/Z_p^j; R) = H_j(X; R)Z_p^j$, the subgroup of $H_j(X; R)$ left fixed by the homology map induced by $\alpha$. (See, for example, [1, p. 37].) Hence our result completely determines the integral homology groups of $X/Z_p^j$.

The proof of Theorem A is based on our paper [4]. We shall briefly summarize here the salient features of that paper: Suppose $R$ is a commutative ring. One may construct from $R$ and the group $Z_p^j$ a ring $\mathfrak{g}$, called the isotopy ring. Suppose $h_t(X, A)$ is an equivariant homology theory defined for pairs of finite CW complexes with cellular $Z_p^j$ actions; assume $h_t(Z_p^j/K) = 0$ for all $i > 0$ and all subgroups $K$ of $Z_p^j$; and that $h_0(Z_p^j/K)$ is an $R$ module for each $K$. Then one may construct a left $\mathfrak{g}$ module $M$ with the following property: For any pair $(X, A)$ of finite CW complexes with cellular $Z_p^j$ action, there is a first quadrant spectral sequence with $E^2_{m,n} = Tor^R_m(GH^n(X, A; R), M)$ which converges to $h^t(X, A)$. Here $Gh^n(X, A; R)$ is a particular right $\mathfrak{g}$ module with the property that, as an $R$ module, $Gh^n(X, A; R) = \oplus H^n(X^K, A^K; R)$, where the summation runs over all subgroups $K$ of $Z_p^j$.

An example. In the proof of Theorem A, it will be convenient to have an example of a $Z_p^j$ action on a $Z_p^s$-sphere. Ultimately, the example will save us some messy algebra in the computation of the $E^2$ terms of various spectral sequences.

Let $\rho_i$ denote the complex numbers with the (linear) action $g \cdot \nu = \exp((2\pi i)/p^{s-j})\nu$, where $g$ generates $Z_p^j$. Then

$$m_0\rho_0 \oplus m_1\rho_1 \oplus \cdots \oplus m_s\rho_s \quad (\text{for } m_i \geq 0)$$

is a vector space of dimension $2(m_0 + m_1 + \cdots + m_s)$ over the reals. Let $X$ denote its unit sphere, so that $X$ is an $n = 2(m_0 + \cdots + m_s) - 1$-sphere. Let $k_i = 2(m_{-i} + m_{-i+1} + \cdots + m_s) - 1$ for $0 \leq i \leq s$. Then $X$ has a $Z_p^j$ action, and $X/Z_p^j$ is a $k$-sphere.

Steenrod and Epstein [3, p. 67] show how to obtain a convenient cell decomposition of $X$ so that $g$ becomes a cellular map. If, for example, the unit sphere $S$ of $\rho_2 \oplus m_1\rho_1 \oplus m_2\rho_2$ has been given a cell decomposition already and $m_2 > 1$, then we obtain a cell decomposition of the unit sphere $T$ of $2\rho_2 \oplus m_1\rho_1 \oplus m_2\rho_2$ as follows: The sphere of $\rho_2$ has a cell decomposition with $p^2$ 0-cells $e^0, ge^0, \ldots, g^{p^2-1}e^0$ and $p^2$ 1-cells $e^1, ge^1, \ldots, g^{p^2-1}e^1$. For the $i$-cells of $T$, $i \leq k = 2(m_{-1} + m_s + 1) - 1$, the dimension of $S$, we use the cells of $S$. $T$ has $p^2(k + 1)$-cells, namely $S \ast g^i e^0$ (the join); and $p^2(k + 2)$-cells, namely $S \ast g^i e^1$.

In this manner we obtain a cell decomposition for $X$ with 1 $i$-cell $e^i$ if $0 \leq i \leq k_0$; with $p$ $i$-cells $e^i, ge^i, \ldots, g^{p-1}e^i$ if $k_0 + 1 \leq i \leq k_1$; \ldots; with $p^m$ $i$-cells $e^i, ge^i, \ldots, g^{p^m-1}e^i$ if $k_{m-1} + 1 \leq i \leq k_m$. It is easy to see that
g(g^ie^i) = g^{j+1}e^i where j + 1 is reduced modulo the relevant power of p. Moreover if i is even and k_{m-1} + 1 < i \leq k_m, then

\[ \partial(g^j e^i) = \sum_{l=0,p^{m-1}} g^l e^{i-1}. \]

If i = k_{m-1} + 1,

\[ \partial(g^j e^i) = \sum_{l=0,\ldots,p^{m-1}} g^l e^{i-1}. \]

If i is odd and k_{m-1} + 1 < i \leq k_m, then

\[ \partial(g^j e^i) = g^{j+1} e^{i-1} - g^j e^{i-1}. \]

Note that by suspending the above X, we may ensure that k_0 be even if desired. We obtain readily the following facts about this X.

**Lemma 1.** Let r \geq s. Suppose 0 \leq k-2 < k_{s-1} < k_s. Then

\[ H_i(X/Z_p^r, X Z_p^{s-1}; Z_p^r) \] equals 0 for i \leq k_{s-1}; Z_p^r for i = k_{s-1} + 1; Z_p^r for k_{s-1} + 2 \leq i \leq k_s.

\[ H_i(X/Z_p^s, X Z_p^{s-2}; Z_p^r) \] equals 0 for i \leq k_{s-2}; Z_p^r for i = k_{s-2} + 1; Z_p^r for k_{s-2} + 2 \leq i < k_{s-1}.

**Proof.** A simple exercise. Q.E.D.

**Proofs.**

**Lemma 2.** Let r \geq s. Suppose X is a finite CW complex with the Z_p^r homology of an n-sphere. Let Z_p^r act cellularly on X, so that X^{Z_p^r} has the Z_p^r homology of a k-sphere, 0 \leq k < n. Then \( H_i(X/Z_p^r, X^{Z_p^r}/Z_p^{s-1}; Z_p^r) \) equals 0 for 0 \leq i < k; Z_p^r for i = k + 1; Z_p^r for k + 2 \leq i < n; Z_p^{s-2} for i = n; 0 for i > n.

**Proof.** Let \( \Gamma \) be the left \( \mathfrak{g} \) module corresponding to the homology theory

\[ h_i(X,A) = H_i(X/Z_p^r, X^{Z_p^r}/Z_p^{s-1} \cup A/Z_p^{s-1}; Z_p^r). \]

There is a spectral sequence with \( E_{a,b}^{2} = \text{Tor}^{\mathfrak{g}}_a (\varphi H_b(X; \mathfrak{g}), \Gamma) \) converging to \( h_i(X) \). Let \( k_i \) be the dimension of \( X^{Z_p^r} \). We assume first that \( k_0 > 0 \). Note that \( E_{a,b}^{2} = 0 \) for \( 0 < b < k_0 \). Hence, for \( b < k_0, E_{b,0}^{2} = h_b(X) \) for any X with the assumed properties. Using Lemma 1, and noting that \( E_{b,0}^{2} \) is independent of \( k_0 \) (as long as \( 0 < k_0 \)), we see \( E_{b,0}^{2} = 0 \) for all \( b \).

Now, since \( E_{a,b}^{2} = 0 \) for \( k_0 < b < k_1 \), it follows \( E_{a,k_0}^{2} = h_{a+k_0}(X) \) for any such X. Using our example, \( E_{a,k_0}^{2} = 0 \) for all \( a \). Continuing in this manner, we see \( E_{a,b}^{2} = 0 \) for \( a < k_{s-1} = k \). But \( E_{a,b}^{2} = 0 \) for \( k < b < k_s = n \). Hence \( E_{a,k}^{2} = h_{a+k}(X) \) for any such X. By Lemma 1, using the independence of \( H_k(X; \mathfrak{g}) \) from \( n \), we see \( E_{a,k}^{2} = 0; E_{b,k}^{2} = Z_p^r; E_{j,k}^{2} = Z_p^r \) for \( i > 2 \). Thus we obtain the lemma for \( i < n \). It is well known that \( h_i(X) = 0 \) for \( i > n \). (See, for example, [1, p. 43].) Finally, \( E_{a,0}^{2} = \varphi H_a(X; \mathfrak{g}) \otimes \Gamma = Z_p^r \), and \( d : E_{a-k+1,0}^{2} \rightarrow E_{a,0}^{2} \) becomes \( d : Z_p^r \rightarrow Z_p^r \). If \( d \) were not one-to-one, then \( h_{n+1}(X) \) would not equal zero. Hence \( E_{n,0}^{\infty} = Z_p^{r-1}, E_{n-k,k}^{\infty} = Z_p^r \), and the order of \( h_n(X) \) is \( p^r \). The case \( r = 1 \) would show that \( h_n(X) = Z_p \). From this
fact, a consideration of cases and the Universal Coefficient Theorem, using the
fact that $h_{n+1}(X) = 0$, shows $h_n(X) = \mathbb{Z}_{p'}$.

Minor modifications in the above argument yield the result if $k_0 = k_1 = \cdots = k_j = -1$ for some $j < s - 1$. Q.E.D.

**Lemma 3.** Let $r > s$. Let $X$ be as in the statement of Theorem A. Suppose $0 < k_{s-2} < k_{s-1} < n$. Let $k = k_{s-2}$. Then $H_i(X/Z_{p^r}, X^{Z_{p^r}}/Z_{p^{r-1}}; \mathbb{Z})$ equals 0 for $0 \leq i \leq k$; $\mathbb{Z}_{p'}$ for $i = k + 1$; $\mathbb{Z}$ for $k + 2 \leq i \leq k_{s-1}$.

**Proof.** The proof is completely analogous to that of Lemma 2, using $h_i(X, A) = H_i(X/Z_{p^r}, X^{Z_{p^r}}/Z_{p^{r-1}}; \mathbb{Z})$, a corresponding left $\mathfrak{g}$ module $\Gamma$, and the fact that

$$E^2_{0,k_{s-1}} = \mathcal{G} H_{k_{s-1}}(X; \mathfrak{g}) \otimes \mathfrak{g} \Gamma = 0.$$ Q.E.D.

**Proof of Theorem A.** We prove Theorem A by induction on $s$. If $s = 1$, we let $\Theta$ be the left $\mathfrak{g}$ module corresponding to $h_i(X, A) = H_i(X/Z_{p^r}, A/Z_{p^r}; \mathbb{Z})$.

The spectral sequence converging to $h_i(X)$ has $E^2_{a,b} = 0$ for $b \neq 0, k_0, k_1$. Assuming $0 < k_0 < k_1 = n$ we see $E^2_{0,0} = h_a(X)$ for any such $X$, if $0 \leq a < k_0$. Hence by use of our example, $E^2_{0,0} = \mathbb{Z}_{p'}$; $E^2_{a,0} = 0$ for $a > 0$. Hence

$$E^2_{a,k_0} = h_{a+k_0}(X)$$

for any such $X$, if $a + k_0 < k_1$. By use of our example,

$$E^2_{0,k_0} = E^1_{1,k_0} = 0; E^2_{a,k_0} = \mathbb{Z}_p$$

for $a \geq 2$. Finally,

$$E^2_{0,k_1} = \mathcal{G} H_{k_1}(X; \mathfrak{g}) \otimes \mathfrak{g} \Gamma = \mathbb{Z}_{p'}.$$

We obtain our result immediately for $0 \leq i < n$ and $i > n$; for the case $i = n$ we argue as at the end of Lemma 2. The case $k_0 \leq 0$ is handled similarly.

We now assume Theorem A for $s - 1$ and prove it for $s$. Hence $H_i(X^{Z_{p^s}}/Z_{p^{s-1}}; \mathbb{Z})$ is known by induction. In particular,

$$H_i(X^{Z_{p^s}}/Z_{p^{s-1}}; \mathbb{Z}) = 0 \quad \text{for } i > k_{s-1}.$$

Yet $H_i(X/Z_{p^s}, X^{Z_{p^s}}/Z_{p^{s-1}}; \mathbb{Z}) = 0$ for $0 \leq i \leq k_{s-1}$ by Lemma 2. The long exact sequence for the pair $(X/Z_{p^s}, X^{Z_{p^s}}/Z_{p^{s-1}})$ then implies

$$H_i(X/Z_{p^s}; \mathbb{Z}) = H_i(X/Z_{p^s}, X^{Z_{p^s}}/Z_{p^{s-1}}; \mathbb{Z}) \quad \text{for } i \geq k_{s-1} + 2$$

and

$$H_i(X/Z_{p^s}; \mathbb{Z}) = H_i(X^{Z_{p^s}}/Z_{p^{s-1}}; \mathbb{Z}) \quad \text{for } 0 \leq i \leq k_{s-1} - 1.$$ This yields our result for all $i$ except $i = k_{s-1}$ and $i = k_{s-1} + 1$.

The same long exact sequence implies that

$$0 = H_{k_{s-1}+1}(X^{Z_{p^s}}/Z_{p^{s-1}}; \mathbb{Z}) \to H_{k_{s-1}+1}(X/Z_{p^s}; \mathbb{Z}) \to \mathbb{Z}_{p'} \to \mathbb{Z}_{p'} \to H_{k_{s-1}}(X/Z_{p^s}; \mathbb{Z}) \to 0$$

is exact. Hence $H_{k_{s-1}+1}(X/Z_{p^s}; \mathbb{Z})$ and $H_{k_{s-1}}(X/Z_{p^s}; \mathbb{Z})$ are both cyclic of the same order.
Assume that $k_{s-2} < k_{s-1}$. Since $p$ is odd, by [2, p. 129] $k_{s-2} \leq k_{s-1} - 2$. Hence the long exact sequence for the pair $(X/Z_{p^s}, XZ_{p^s}/Z_{p^{s-2}})$ yields an exact sequence

$$0 = H_{k_{s-1}}(XZ_{p^s}/Z_{p^{s-2}}; Z_{p^s}) \rightarrow H_{k_{s-1}}(X/Z_{p^s}; Z_{p^s})$$

$$\rightarrow H_{k_{s-1}}(X/Z_{p^s}, XZ_{p^s}/Z_{p^{s-2}}; Z_{p^s}) \rightarrow H_{k_{s-1}}(XZ_{p^s}/Z_{p^{s-2}}; Z_{p^s}) = 0.$$ 

Hence, by Lemma 3,

$$H_{k_{s-1}}(X/Z_{p^s}; Z_{p^s}) = Z_{p^{s-1}}$$

and our theorem follows.

If $k_{s-2} = k_{s-1}$ but $k_{s-3} < k_{s-1}$, one deals similarly with the exact sequence of the pair $X/Z_{p^s}, XZ_{p^s}/Z_{p^{s-3}}$. The general result is clear.

If $p = 2$ and $k_{s-2} = k_{s-1} - 1$, then we obtain the exact sequence

$$0 \rightarrow H_{k_{s-1}}(X/Z_{p^s}; Z_{p^s}) \rightarrow Z_{p^s} \rightarrow H_{k_{s-2}}(X/Z_{p^s}; Z_{p^s})$$

$$\rightarrow Z_{p^s} \rightarrow H_{k_{s-2}}(X/Z_{p^s}; Z_{p^s}) = Z_{p^{s-2}} \rightarrow 0.$$ 

Hence, in this case,

$$H_{k_{s-1}}(X/Z_{p^s}; Z_{p^s}) = Z_{p^{s-2}} = H_{k_{s-2}}(X/Z_{p^s}; Z_{p^s}).$$

Q.E.D.

References


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