A NOTE ON REGULAR METHODS OF SUMMABILITY AND THE BANACH-SAKS PROPERTY

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Abstract. Using the Galvin-Prikry partition theorem from set theory it is proved that every bounded sequence in a Banach space has a subsequence such that either every subsequence of which is summable or no subsequence of which is summable.

The infinite matrix \( \{a_{ij}\}_{i \in \omega, j \in \omega} \) (\( \omega \) is the set of natural numbers) is called a regular method of summability if given a sequence \( \langle e_i \rangle_{i \in \omega} \) of elements of a Banach space \( B \), converging in norm to \( e \), then the sequence \( e'_i = \sum_{j=0}^{\infty} a_{ij} e_j \) converges also to \( e \). The sequence \( \langle e_i \rangle_{i \in \omega} \) is called summable with respect to \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \) if \( e'_i = \sum_{j=0}^{\infty} a_{ij} e_j \) converges in norm. (See [2, p. 75] for reference.) It is well known [2] that \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \) is a regular method of summability if and only if

(a) l.u.b. \( \sum_{j=0}^{\infty} |a_{ij}| < M < \infty, \)
(b) \( \lim_{i \to \infty} a_{ij} = 0 \) for every \( j, \)
(c) \( \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ij} = 1. \)

In this note we prove:

Theorem. Let \( \langle e_i \rangle_{i \in \omega} \) be a bounded sequence of elements in a Banach space \( B \), and \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \) a regular method of summability; then there exists a subsequence of \( \langle e_i \rangle_{i \in \omega} \), \( \langle e_{ik} \rangle_{k \in \omega} \), such that:

(a) every subsequence of \( \langle e_{ik} \rangle_{k \in \omega} \) is summable with respect to \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \), each being summed to the same limit; or
(b) no subsequence of \( \langle e_{ik} \rangle_{k \in \omega} \) is summable with respect to \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \).

Proof. Let \( P(\omega) \) be the set of all infinite subsets of \( \omega \). There exists a natural topology on \( P(\omega) \) generated by the subbasis \( \{A_n\}_{n \in \omega} \cup \{B_n\}_{n \in \omega} \) where

\[ A_n = \{p \mid p \in P(\omega), n \in p\}, \quad B_n = \{p \mid p \in P(\omega), n \notin p\}. \]

Define a partition of \( P(\omega) \) into two Borel sets:

\[ A = \{p \mid \langle e_i \rangle_{i \in p} \text{ is summable w.r.t. } \langle a_{ij} \rangle_{i \in \omega, j \in \omega}\}, \]
\[ B = P(\omega) - A. \]

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((e_i)_{i\in\omega} is the subsequence of \((e_i)_{i\in\omega}\) obtained by enumerating e_i for i \in p in the natural order of the i's).

We prove that A is a Borel subset of \(P(\omega)\). Let

\[
B_{e,m,n} = \left\{ p \left\| \sum_{j=0}^{\infty} a_{nj} \cdot e_{kj} - \sum_{j=0}^{\infty} a_{mj} e_{kj} \right\| < \epsilon \right\}
\]

where \(k_j\) is a monotone enumeration of \(p\).

\(B_{e,m,n}\) is open in our topology on \(P(\omega)\), because if \(p \in B_{e,m,n}\), pick \(\epsilon'\) such that

\[
\left\| \sum_{j=0}^{\infty} a_{mj} e_{kj} - \sum_{j=0}^{\infty} a_{nj} e_{kj} \right\| < \epsilon' < \epsilon.
\]

Let \(J\) be large enough such that

\[
T\left( \sum_{j=0}^{\infty} |a_{mj}| + \sum_{j=0}^{\infty} |a_{nj}| \right) < \frac{\epsilon - \epsilon'}{2}
\]

where \(T\) is a bound for \(\|e_i\|\). (\(J\) exists because \((a_{ij})_{i\in\omega,j\in\omega}\) is a regular method of summability.)

The set \(C = \{ q | q \in P(\omega), q \cap \{l|l < J\} = p \cap \{l|l < J\} \}\) is an open subset of \(P(\omega)\), \(p \in C\) and \(C \subseteq B_{e,m,n}\). This last inclusion is true since if \(q \in C\) and \(l_j\) is a monotone enumeration of \(q\), then \(l_j = k_j\) for \(j < J\). Hence,

\[
\left\| \sum_{j=0}^{\infty} a_{mj} e_{lj} - \sum_{j=0}^{\infty} a_{nj} e_{lj} \right\|
\]

\[
\leq \left\| \sum_{j=0}^{J-1} a_{mj} e_{lj} - \sum_{j=0}^{J-1} a_{nj} e_{lj} \right\| + \left\| \sum_{j=0}^{\infty} a_{mj} e_{lj} - \sum_{j=0}^{\infty} a_{nj} e_{lj} \right\|
\]

\[
\leq \left\| \sum_{j=0}^{J-1} a_{mj} e_{lj} - \sum_{j=0}^{J-1} a_{nj} e_{lj} \right\| + T\left( \sum_{j=0}^{\infty} |a_{mj}| + \sum_{j=0}^{\infty} |a_{nj}| \right)
\]

\[
< \epsilon' + T \left( \sum_{j=0}^{\infty} |a_{mj}| + \sum_{j=0}^{\infty} |a_{nj}| \right) + \frac{\epsilon - \epsilon'}{2}
\]

\[
< \epsilon' + \frac{(\epsilon - \epsilon')/2 + (\epsilon - \epsilon')/2} = \epsilon.
\]

Thus every element of \(B_{e,m,n}\) has an open neighborhood included in \(B_{e,m,n}\). Hence \(B_{e,m,n}\) is open.

The set \(A\) is \(\cap_{k} \bigcup_{N} \bigcap_{n \geq N} B_{l/k,m,n}\). \(A\) is the set of those \(p\) such that \(\sum_{j=0}^{\infty} a_{j} e_{kj}\) is a Cauchy sequence if \(k_j\) is a monotone enumeration of \(p\). By a theorem of F. Galvin and K. Prikry [3] there is \(q \in P(\omega)\) such that either

(I) for every \(t \subseteq q\), \(t \in P(\omega) \Rightarrow t \in A\), or

(II) for every \(t \subseteq q\), \(t \in P(\omega) \Rightarrow t \in B\).

For the sequence \(\langle e_i \rangle_{i \in q}\) either (b) holds (in case (II)) or in case (I) we shall indicate how to pick a subsequence of it for which (a) holds. If we assume that

(I) holds, then every subsequence of \(\langle e_i \rangle_{i \in q}\) is summable to a limit which lies
in the subspace spanned by $\langle e_i \rangle_{i \in q}$. Call it $B'$, which is of course separable. For every $n \in \omega$, $n \neq 0$, let $\{A^m_n | m \in \omega\}$ be a family of open balls of radius $1/n$ covering $B'$. By induction we get a sequence $\cdots \subseteq q_3^1 \subseteq q_2^1 \subseteq q_1^1 \subseteq q$ such that either (A) every subsequence of $\langle e_i \rangle_{i \in q_k^1}$ is summable to a limit in $A_k^1$ or (B) every subsequence of $\langle e_i \rangle_{i \in q_k^1}$ is summable to a limit which is outside $A_k^1$. (We can get the $q_{k+1}^1$ from $q_k^1$ by again using the Galvin-Prikry result, noting as before that the partition of $P_x(q_k)$ is Borel.) Clearly for some $k_1$ we get (A) to hold. Let $q_{k_1}^1$ be elements of the diagonal sequence of the natural enumerations of $q_k^1$. Now get $\cdots \subseteq q_2^1 \subseteq q_1^1 \subseteq q_\infty$ such that either (A): every subsequence of $\langle e_i \rangle_{i \in q_k^1}$ is summable to a limit in $A_k^2$ or (B): every subsequence of $\langle e_i \rangle_{i \in q_k^1}$ is summable to a limit outside $A_k^2$. Again we get $k_2$ for which (A) holds. $q_{k_2}^2$, $q_{k_3}^2$, etc., and $k_1$, $k_2$, $k_3$, ... are defined as before. Let $t$ be the set of elements of the diagonal sequence of the sequence generated by the $q_k^n$. Every subsequence of $\langle e_i \rangle_{i \in t}$ is summable to a limit which is in $A_{k_n}^n$, for every $n$ hence to a limit in $A_{k_n}^n$ which contains at most one point. Hence the sequence $\langle e_i \rangle_{i \in t}$ satisfies (a).

REMARKS. (1) By using the theorem countably many times (using the fact that finitely many changes in a sequence do not influence its summability), we can get the conclusion to hold simultaneously for a countable sequence of regular summability methods such that the limit for those of them for which (1) holds is the same.

(2) A Banach space is said to have the Banach-Saks property with respect to the regular method of summability $\langle a_{ij} \rangle_{i, j \in \omega}$ if every bounded sequence has a summable subsequence. (See [1]. The problem solved by this note is due to Louis Sucheston.) As a corollary to the theorem we get: If $B$ has the Banach-Saks property with respect to the regular method of summability $\langle a_{ij} \rangle_{i \in \omega, j \in \omega}$, then every bounded sequence has a subsequence such that each of its subsequences is summable with respect to $\langle a_{ij} \rangle_{i \in \omega, j \in \omega}$.

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