A NOTE ON REGULAR METHODS OF SUMMABILITY AND THE BANACH-SAKS PROPERTY

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Abstract. Using the Galvin-Prikry partition theorem from set theory it is proved that every bounded sequence in a Banach space has a subsequence such that either every subsequence of which is summable or no subsequence of which is summable.

The infinite matrix \( \{a_{ij}\}_{i \in \omega, j \in \omega} \) (\( \omega \) is the set of natural numbers) is called a regular method of summability if given a sequence \( \langle e_i \rangle_{i \in \omega} \) of elements of a Banach space \( B \), converging in norm to \( e \), then the sequence \( e'_i = \sum_{j=0}^{\infty} a_{ij} e_j \) converges also to \( e \). The sequence \( \langle e_i \rangle_{i \in \omega} \) is called summable with respect to \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \) if \( e'_i = \sum_{j=0}^{\infty} a_{ij} e_j \) converges in norm. (See [2, p. 75] for reference.)

It is well known [2] that \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \) is a regular method of summability if and only if

(a) \( \text{l.u.b.} \sum_{j=0}^{\infty} |a_{ij}| < M < \infty \),
(b) \( \lim_{i \to \infty} a_{ij} = 0 \) for every \( j \),
(c) \( \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ij} = 1 \).

In this note we prove:

Theorem. Let \( \langle e_i \rangle_{i \in \omega} \) be a bounded sequence of elements in a Banach space \( B \), and \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \) a regular method of summability; then there exists a subsequence of \( \langle e_i \rangle_{i \in \omega} \), \( \langle e_{ik} \rangle_{k \in \omega} \) such that:

(a) every subsequence of \( \langle e_{ik} \rangle_{k \in \omega} \) is summable with respect to \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \), each being summed to the same limit; or

(b) no subsequence of \( \langle e_{ik} \rangle_{k \in \omega} \) is summable with respect to \( \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \).

Proof. Let \( P(\omega) \) be the set of all infinite subsets of \( \omega \). There exists a natural topology on \( P(\omega) \) generated by the subbasis \( \{A_n \}_{n \in \omega} \cup \{B_n \}_{n \in \omega} \) where

\[ A_n = \{ p \mid p \in P(\omega), n \in p \}, \quad B_n = \{ p \mid p \in P(\omega), n \notin p \}. \]

Define a partition of \( P(\omega) \) into two Borel sets:

\[ A = \{ p \mid \langle e_i \rangle_{i \in p} \text{ is summable w.r.t. } \langle a_{ij} \rangle_{i \in \omega, j \in \omega} \}, \]
\[ B = P(\omega) - A \]
We prove that $A$ is a Borel subset of $P(\omega)$. Let

$$B_{e,m,n} = \left\{ p \mid \sum_{j=0}^{\infty} a_{nj} \cdot e_{kj} - \sum_{j=0}^{\infty} a_{mj} e_{kj} \right\} < \varepsilon$$

where $k_j$ is a monotone enumeration of $p$.

$B_{e,m,n}$ is open in our topology on $P(\omega)$, because if $p \in B_{e,m,n}$, pick $\varepsilon'$ such that

$$\left\| \sum_{j=0}^{\infty} a_{mj} e_{kj} - \sum_{j=0}^{\infty} a_{nj} e_{kj} \right\| < \varepsilon'< \varepsilon.$$ 

Let $J$ be large enough such that

$$T\left( \sum_{j=1}^{\infty} |a_{mj}| + \sum_{j=1}^{\infty} |a_{nj}| \right) < \varepsilon - \varepsilon'$$

where $T$ is a bound for $\|e_i\|$. ($T$ exists because $\langle a_{ij} \rangle_{i \in \omega, j \in \omega}$ is a regular method of summability.)

The set $C = \{ q \mid q \in P(\omega), q \cap \{ l \mid l < J \} = p \cap \{ l \mid l < J \} \}$ is an open subset of $P(\omega)$. $p \in C$ and $C \subseteq B_{e,m,n}$. This last inclusion is true since if $q \in C$ and $l_j$ is a monotone enumeration of $q$, then $l_j = k_j$ for $j < J$. Hence,

$$\left\| \sum_{j=0}^{\infty} a_{mj} e_{lj} - \sum_{j=0}^{\infty} a_{nj} e_j \right\| < \varepsilon' + T \cdot \left( \sum_{j=1}^{\infty} |a_{mj}| + \sum_{j=1}^{\infty} |a_{nj}| \right) + \varepsilon - \varepsilon'$$

Thus every element of $B_{e,m,n}$ has an open neighborhood included in $B_{e,m,n}$. Hence $B_{e,m,n}$ is open.

The set $A$ is $\bigcap_k \bigcup_N \bigcap_{m,n>N} B_{l/k,m,n}$. ($A$ is the set of those $p$ such that $\sum_{j=0}^{\infty} a_j e_{kj}$ is a Cauchy sequence if $k_j$ is a monotone enumeration of $p$.) By a theorem of F. Galvin and K. Prikry [3] there is $q \in P(\omega)$ such that either

(I) for every $t \subseteq q$, $t \in P(\omega) \Rightarrow t \in A$, or

(II) for every $t \subseteq q$, $t \in P(\omega) \Rightarrow t \in B$.

For the sequence $\langle e_i \rangle_{i \in q}$ either (b) holds (in case (II)) or in case (I) we shall indicate how to pick a subsequence of it for which (a) holds. If we assume that (I) holds, then every subsequence of $\langle e_i \rangle_{i \in q}$ is summable to a limit which lies
in the subspace spanned by $\langle e_i \rangle_{i \in \mathbb{N}}$. Call it $B'$, which is of course separable. For every $n \in \mathbb{N}$, $n \neq 0$, let $\langle A^m \mid m \in \mathbb{N} \rangle$ be a family of open balls of radius $1/n$ covering $B'$. By induction we get a sequence $\cdots \subseteq q_3 \subseteq q_2 \subseteq q_1 \subseteq q$ such that either (A) every subsequence of $\langle e_i \rangle_{i \in q^1}$ is summable to a limit in $A_k^1$ or (B) every subsequence of $\langle e_i \rangle_{i \in q^1}$ is summable to a limit which is outside $A_k^1$. (We can get the $q^1_{k+1}$ from $q_k$ by again using the Galvin-Prikry result, noting as before that the partition of $P(q_k)$ is Borel.) Clearly for some $k_1$ we get (A) to hold. Let $q^1_{\infty}$ be elements of the diagonal sequence of the natural enumerations of $q^1_k$. Now get $\cdots \subseteq q^2_3 \subseteq q^2_2 \subseteq q^1_1 \subseteq q^1_{\infty}$ such that either (A): every subsequence of $\langle e_i \rangle_{i \in q^2_k}$ is summable to a limit in $A_k^2$ or (B): every subsequence of $\langle e_i \rangle_{i \in q^2_k}$ is summable to a limit outside $A_k^2$. Again we get $k_2$ for which (A) holds. $q^2_3$, $q^2_2$, etc., and $k_1$, $k_2$, $k_3$, ... are defined as before. Let $t$ be the set of elements of the diagonal sequence of the sequence generated by the $q^k_n$. Every subsequence of $\langle e_i \rangle_{i \in t}$ is summable to a limit which is in $A_k^n$, for every $n$ hence to a limit in $A_k^n$, which contains at most one point. Hence the sequence $\langle e_i \rangle_{i \in \mathbb{N}}$ satisfies (a).

REMARKS. (1) By using the theorem countably many times (using the fact that finitely many changes in a sequence do not influence its summability), we can get the conclusion to hold simultaneously for a countable sequence of regular summability methods such that the limit for those of them for which (I) holds is the same.

(2) A Banach space is said to have the Banach-Saks property with respect to the regular method of summability $\langle a_y \rangle_{i \in \mathbb{N}}$ if every bounded sequence has a summable subsequence. (See [1]. The problem solved by this note is due to Louis Sucheston.) As a corollary to the theorem we get: If $B$ has the Banach-Saks property with respect to the regular method of summability $\langle a_y \rangle_{i \in \mathbb{N}, j \in \mathbb{N}}$, then every bounded sequence has a subsequence such that each of its subsequences is summable with respect to $\langle a_y \rangle_{i \in \mathbb{N}, j \in \mathbb{N}}$.

REFERENCES


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