AN APPLICATION OF THEOREMS OF SCHUR AND ALBERT

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Abstract. Suppose $\Pi_n$ is the cone of $n \times n$ positive semidefinite matrices, and $\text{int}(\Pi_n)$ is the set of positive definite matrices. Theorems of Schur and Albert are applied to obtain some elements of $\Pi_n$ and $\text{int}(\Pi_n)$. Then an analogue of Albert's theorem is given for $M$-matrices, and finally a generalization is given for matrices of class $P$.

I. Introduction. Suppose $\Pi_n$ is the cone of $n \times n$ positive semidefinite matrices over the complex field. The interior of $\Pi_n$, denoted $\text{int}(\Pi_n)$, is the set of $n \times n$ positive definite matrices.

If $A$ and $B$ are arbitrary matrices of the same size, the Hadamard product of $A$ and $B$ is the matrix $A*B$ whose $(i,j)$ entry is $a_{ij}b_{ij}$. A rather comprehensive account of this product is given in [9].

J. Schur proved the following theorem.

Theorem 1.1 [8]. If $A, B \in \Pi_n$, then $A*B \in \Pi_n$. Further, if $A, B \in \text{int}(\Pi_n)$, then $A*B \in \text{int}(\Pi_n)$.

This theorem is easily proved by noting $A*B$ is a principal submatrix of the tensor product of $A$ and $B$.

Now suppose $M$ is a matrix partitioned in the form

\begin{equation}
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\end{equation}

In [2], the generalized Schur complement of $A$ in $M$ is defined as

\begin{equation}
M|A = D - CA^+B,
\end{equation}

where $A^+$ is the Moore-Penrose inverse of $A$. Similarly, we define

\begin{equation}
M|D = A - BD^+C.
\end{equation}

If $M$ given in (1.1) is hermitian and is partitioned symmetrically, then $C = B^*$. For this case, Albert [1] has proved the following theorem, which was generalized in [2, Theorem 2].

Theorem 1.2. Suppose $M$ is hermitian and partitioned symmetrically in (1.1). Then $M \in \Pi_n$ if and only if $A \in \Pi_k$, $M|A \in \Pi_{n-k}$ and the null space of $A$ is
contained in the null space of $B^*$ (i.e. $N(A) \subseteq N(B^*)$). Further, $M \in \text{int}(\Pi_n)$ if and only if $A \in \text{int}(\Pi_k)$, $M|A \in \text{int}(\Pi_{n-k})$, and $M|D \in \text{int}(\Pi_k)$.

We shall utilize Theorems 1.1 and 1.2 to obtain some new results on positive semidefinite matrices.

II. Some elements of $\Pi_n$. As in §1, $N(A)$ will denote the null space of the matrix $A$.

**Theorem 2.** Suppose each of $A, B, C, D$ is in $\Pi_n$, and $N(A) \subseteq N(B)$, $N(C) \subseteq N(D)$. Then

$$BA^*B + DC^*D - (B^*D)(A^*C)^+ (B^*D) \in \Pi_n.$$  

**Proof.** Let

$$M = \begin{pmatrix} A & B \\ B & BA^*B \end{pmatrix}, \quad N = \begin{pmatrix} C & D \\ D & DC^*D \end{pmatrix}.$$  

Both $M$ and $N$ are in $\Pi_{2n}$ by Albert’s theorem. Then applying Schur’s theorem, we get

$$(2.1) \quad M \ast N = \begin{pmatrix} A \ast C & B \ast D \\ B \ast D & (BA^*B) \ast (DC^*D) \end{pmatrix} \in \Pi_{2n}.$$  

Now we reapply Theorem 1.2 to (2.1) and obtain $(BA^*B) \ast (DC^*D) - (B^*D)(A^*C)^+ (B^*D) \in \Pi_n$. □

Note that as a consequence of Theorem 1.2, using the assumptions of the above theorem, we obtain that $N(A \ast C) \subseteq N(B \ast D)^*$.

One can obtain readily now a number of corollaries; we shall mention a few of these.

**Corollary 2.1.** If $A, C \in \text{int}(\Pi_n)$, then $A^{-1} \ast C^{-1} - (A \ast C)^{-1} \in \Pi_n$.

**Proof.** Let $B = I_n = D$ in Theorem 2, and use the fact that $A^* = A^{-1}$ if $A$ is invertible.

**Corollary 2.2.** Suppose $A, B \in \text{int}(\Pi_n)$; $C, D \in \Pi_n$. Then $(A \ast B^{-1} + C) - (A^{-1} \ast B + D)^{-1} \in \Pi_n$.

**Proof.** As in the proof of Theorem 2, let

$$M = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} B^{-1} & I \\ I & B \end{pmatrix}$$  

and put

$$P = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}.$$  

Then $M \ast N + P \in \Pi_{2n}$, and the result follows by the technique used previously.

From Corollary 2.2, one obtains immediately the result that if $C, D \in \Pi_n$, then $(I + C) - (I + D)^{-1} \in \Pi_n$. Simply choose $A = B = I_n$ above.

**Corollary 2.3.** Let $A \in \text{int}(\Pi_n)$. Then
THEOREMS OF SCHUR AND ALBERT

\[ A \ast A - (A \ast I)(A^{-1} \ast A + I)^{-1}(A \ast I) \]

is in \( \Pi_n \).

**Proof.** Let
\[
M = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} A & A \\ A & A \end{pmatrix}, \quad \text{and} \quad P = \begin{pmatrix} 0 & I_n \\ 0 & 1_n \end{pmatrix}.
\]

Then \( M \ast N + P \in \Pi_{2n} \), and the result follows as in the previous corollary.

In fact, even more is known concerning Corollary 2.3. In [9, Corollary 4.3, p. 236], Styan shows that \( A \ast A - 2(A \ast I)(A^{-1} \ast A + I)^{-1}(A \ast I) \in \Pi_n \) using a technique based on probabilistic methods.

We also would like to point out that Theorem 2 is an analogue for the Schur product of Theorem 5 of [2]. There it is shown that if \( A, C \in \Pi_n \), and if \( B, D \) are chosen so that \( N(A) \subseteq N(B^*) \), \( N(C) \subseteq N(D^*) \), then
\[
B^*A + B + D^*C + D - (B + D)^*(A + C) + (B + D) \in \Pi_n.
\]

From Corollary 2.2, if \( A, B \in \text{int}(\Pi_n) \), then it follows that \( A \ast B - (A^{-1} \ast B^{-1})^{-1} \in \Pi_n \). There is an analogue of this result for matrix addition, i.e. \( A + B - (A^{-1} + B^{-1})^{-1} \in \text{int}(\Pi_n) \). This is a consequence of the previously mentioned result of Carlson, Haynsworth and Markham [2]; we offer a simple proof of this fact.

Let
\[
M = \begin{pmatrix} A & \frac{1}{2}I \\ \frac{1}{2} & A^{-1} \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} B & \frac{1}{2}I \\ \frac{1}{2}I & B^{-1} \end{pmatrix}.
\]

By Theorem 1.2, both \( M \) and \( N \) belong to \( \text{int}(\Pi_{2n}) \). Now
\[
M + N = \begin{pmatrix} A + B & I \\ I & A^{-1} + B^{-1} \end{pmatrix} \in \text{int}(\Pi_{2n}).
\]

Apply Theorem 1.2 again. Then \( M + N[A^{-1} + B^{-1}] \in \text{int}(\Pi_n) \). But \( M + N[A^{-1} + B^{-1}] = A + B - (A^{-1} + B^{-1})^{-1} \).

**III.** \( M \)-matrices. Suppose \( A \) is a square matrix over the real field. Let \( Z_n \) denote the class of \( n \times n \) matrices whose off-diagonal entries are nonpositive. Assume \( A \in Z_n \). \( A \) is called an \( M \)-matrix, see [6], if and only if \( A \) is invertible and \( A^{-1} \) is a nonnegative matrix (each entry is nonnegative). Let
\[
G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

where \( A \) and \( D \) are square matrices of order \( k \) and \( n - k \), respectively.

If \( G \) is an \( M \)-matrix, then it is well known that \( A \) and \( D \) are \( M \)-matrices. Fan [5] proved that if \( D \) has order 1, then \( G|D \) is an \( M \)-matrix. Crabtree [3, Lemma 1] extended this result to \( D \) of arbitrary order. Watford [10], in turn, proved this result for generalized \( M \)-matrices with respect to a cone.
These results are useful in obtaining an analogue of Albert's Theorem 1.2 for \( M \)-matrices.

**Theorem 3.** Suppose \( G \) is an \( n \times n \) matrix partitioned as in (3.1), and \( G \) is in \( Z_n \). Then \( G \) is an \( M \)-matrix if and only if \( A, D, G|A, \) and \( G|D \) are \( M \)-matrices.

**Proof.** If \( G \) is an \( M \)-matrix, then \( A, D, G|A, \) and \( G|D \) are \( M \)-matrices by the comments preceding Theorem 3.

Now suppose \( A, D, G|A, G|D \) are \( M \)-matrices. Let

\[
\bar{G} = \begin{bmatrix} (G|D)^{-1} & -A^{-1}B(G|A)^{-1} \\ -D^{-1}C(G|D)^{-1} & (G|A)^{-1} \end{bmatrix}.
\]

It is easy to verify \( G \cdot \bar{G} = I \), so \( G^{-1} \) exists. Further, \( G^{-1} \) is nonnegative since each of \( A^{-1}, D^{-1}, (G|A)^{-1}, \) and \( (G|D)^{-1} \) is nonnegative, and \( B \) and \( C \) are nonpositive. Thus \( G \) is an \( M \)-matrix. \( \square \)

Theorem 3 offers a practical procedure for determining if a given matrix is an \( M \)-matrix.

Now we will take a closer look at Albert's theorem. First, we need some additional notation. If \( \alpha \) and \( \beta \) are strictly increasing sequences on \( \{1, 2, \ldots, n\} \) of the same length, then \( M(\alpha|\beta) \) will denote the minor of \( M \) with rows indexed by \( \alpha \) and columns indexed by \( \beta \). If \( \alpha = \beta \), then we write \( M(\alpha) \).

If \( M \) is partitioned as in (1.1), where \( A \) is nonsingular of order \( k \), then

\[
M|A = (e_{ij}), i, j = k + 1, \ldots, n,
\]

with

\[
e_y = \frac{M(1, 2, \ldots, k, i|1, 2, \ldots, k, j)}{M(1, 2, \ldots, k)} = \frac{M(1, 2, \ldots, k, i|1, 2, \ldots, k, j)}{\det(A)};
\]

see [4].

If \( M \) is hermitian, then \( M \) is positive definite if and only if the leading principal minors of \( M \) are positive. Hence we can rephrase Albert's theorem for this case.

**Theorem 4.** Suppose \( M \) is hermitian, and is partitioned symmetrically in (1.1). Then \( M \in \text{int}(\Pi_n) \) if and only if \( A \in \text{int}(\Pi_k) \) and \( M|A \in \text{int}(\Pi_{n-k}) \).

**Proof.** It is well known that if \( M \in \text{int}(\Pi_n) \), then \( A \) and \( M|A \) are positive definite.

Conversely, we need only show that the leading principal minors of \( M \) are positive. Consider an arbitrary minor, say \( M(1, \ldots, i) \). If \( i < k \), this minor is positive since it is a principal minor of \( A \). Assume \( i > k \). Then, using an identity of Sylvester [7, p. 101], we have

\[
M|A(k + 1, \ldots, i) = (\det(A))^{-1}M(1, \ldots, k + 1, \ldots, i).
\]

The result now follows. \( \square \)

If \( M \in Z \), then \( M \) is an \( M \)-matrix if and only if the leading principal
minors of $M$ are positive. Thus, Theorem 3 could also be restated in the form of Theorem 4.

**Definition [6].** Suppose $M$ is an $n \times n$ matrix. Then $M$ belongs to class $P$ if and only if all principal minors of $M$ are positive.

We can generalize Albert's theorem to class $P$ in the following manner.

**Theorem 5.** Let $M$ be partitioned as in (1.1), where the submatrix $A$ has order $1$. Then

$$(3.3) \quad M \in P \text{ if and only if } A \in P, M \mid A \in P, \text{ and } D \in P.$$ 

We omit the proof since the techniques are similar to those of Theorem 4.

Observe the following concerning Theorem 5. On the one hand, to see if $M \in P$, there are $2^n - 1$ principal minors to check. Applying the above result, we obtain a number and two matrices of order $n - 1$ to check the principal minors. Using this equivalence iteratively (to the right-hand side of (3.3)), there are $1 + 2 + \cdots + 2^n - 1$ numbers which must be verified to be positive. But $1 + 2 + \cdots + 2^n - 1 = 2^n - 1$ for $n$ a positive integer, so, in fact, the same number of elements must be verified. The obvious advantage of the right-hand side of (3.3) lies in the reduction of the order of the matrices at each iteration.

It is possible to reduce the number of minors checked? For example, if $M$ has leading positive principal minors, then $M$ does not necessarily belong to class $P$. A simple example to illustrate is $M = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Does there exist an analogue to Theorem 3 for class $P$ when $M$ is partitioned as in (1.1), with $A$ of order $k$? If $M$ has order 2 or 3, the result holds. For larger orders, it need not hold. Consider

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 3 & 5 & 1 & 2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

Here $A$, $M \mid A$, $D$, and $M \mid D$ are all in class $P$, but $M(13)$ is zero.

We conclude with the following query. Suppose $M$ is an $n \times n$ matrix. What is the minimal number of principal minors of $M$ that must be positive in order that $M$ belong to class $P$? Is it necessary to verify that all $2^n - 1$ principal minors, or related minors, are positive?

**References**


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