AN APPLICATION OF THEOREMS OF SCHUR AND ALBERT

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Abstract. Suppose \( \Pi_n \) is the cone of \( n \times n \) positive semidefinite matrices, and \( \text{int}(\Pi_n) \) is the set of positive definite matrices. Theorems of Schur and Albert are applied to obtain some elements of \( \Pi_n \) and \( \text{int}(\Pi_n) \). Then an analogue of Albert's theorem is given for \( M \)-matrices, and finally a generalization is given for matrices of class \( P \).

I. Introduction. Suppose \( \Pi_n \) is the cone of \( n \times n \) positive semidefinite matrices over the complex field. The interior of \( \Pi_n \), denoted \( \text{int}(\Pi_n) \), is the set of \( n \times n \) positive definite matrices.

If \( A \) and \( B \) are arbitrary matrices of the same size, the Hadamard product of \( A \) and \( B \) is the matrix \( A \ast B \) whose \((i,j)\) entry is \( a_{ij}b_{ij} \). A rather comprehensive account of this product is given in [9].

J. Schur proved the following theorem.

Theorem 1.1 [8]. If \( A, B \in \Pi_n \), then \( A \ast B \in \Pi_n \). Further, if \( A, B \in \text{int}(\Pi_n) \), then \( A \ast B \in \text{int}(\Pi_n) \).

This theorem is easily proved by noting \( A \ast B \) is a principal submatrix of the tensor product of \( A \) and \( B \).

Now suppose \( M \) is a matrix partitioned in the form

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

In [2], the generalized Schur complement of \( A \) in \( M \) is defined as

\[
M \vert A = D - CA^+B,
\]

where \( A^+ \) is the Moore-Penrose inverse of \( A \). Similarly, we define

\[
M \vert D = A - BD^+C.
\]

If \( M \) given in (1.1) is hermitian and is partitioned symmetrically, then \( C = B^* \). For this case, Albert [1] has proved the following theorem, which was generalized in [2, Theorem 2].

Theorem 1.2. Suppose \( M \) is hermitian and partitioned symmetrically in (1.1). Then \( M \in \Pi_n \) if and only if \( A \in \Pi_k, M \vert A \in \Pi_{n-k} \) and the null space of \( A \) is...
contained in the null space of $B^*$ (i.e. $N(A) \subseteq N(B^*)$). Further, $M \in \text{int}(\Pi_n)$ if and only if $A \in \text{int}(\Pi_k)$, $M|A \in \text{int}(\Pi_{n-k})$, and $M|D \in \text{int}(\Pi_k)$.

We shall utilize Theorems 1.1 and 1.2 to obtain some new results on positive semidefinite matrices.

II. Some elements of $\Pi_n$. As in §I, $N(A)$ will denote the null space of the matrix $A$.

THEOREM 2. Suppose each of $A$, $B$, $C$, $D$ is in $\Pi_n$, and $N(A) \subseteq N(B)$, $N(C) \subseteq N(D)$. Then

$$BA^*B \cdot DC^*D - (B \cdot D)(A \cdot C)^+ (B \cdot D) \in \Pi_n.$$ 

PROOF. Let

$$M = \begin{pmatrix} A & B \\ B & BA^*B \end{pmatrix}, \quad N = \begin{pmatrix} C & D \\ D & DC^*D \end{pmatrix}.$$ 

Both $M$ and $N$ are in $\Pi_{2n}$ by Albert's theorem. Then applying Schur's theorem, we get

$$(2.1) \quad M \cdot N = \begin{pmatrix} A \cdot C & B \cdot D \\ B \cdot D & (BA^*B) \cdot (DC^*D) \end{pmatrix} \in \Pi_{2n}.$$ 

Now we reapply Theorem 1.2 to (2.1) and obtain $(BA^*B) \cdot (DC^*D) - (B \cdot D)(A \cdot C)^+ (B \cdot D) \in \Pi_n$. □

Note that as a consequence of Theorem 1.2, using the assumptions of the above theorem, we obtain that $N(A \cdot C) \subseteq N(B \cdot D)^*$.

One can obtain readily now a number of corollaries; we shall mention a few of these.

COROLLARY 2.1. If $A, C \in \text{int}(\Pi_n)$, then $A^{-1} \cdot C^{-1} - (A \cdot C)^{-1} \in \Pi_n$.

PROOF. Let $B = I_n = D$ in Theorem 2, and use the fact that $A^* = A^{-1}$ if $A$ is invertible.

COROLLARY 2.2. Suppose $A, B \in \text{int}(\Pi_n)$; $C, D \in \Pi_n$. Then $(A \cdot B^{-1} + C) - (A^{-1} \cdot B + D)^{-1} \in \Pi_n$.

PROOF. As in the proof of Theorem 2, let

$$M = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} B^{-1} & I \\ I & B \end{pmatrix}$$ 

and put

$$P = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}.$$ 

Then $M \cdot N + P \in \Pi_{2n}$, and the result follows by the technique used previously.

From Corollary 2.2, one obtains immediately the result that if $C, D \in \Pi_n$, then $(I + C) - (I + D)^{-1} \in \Pi_n$. Simply choose $A = B = I_n$ above.

COROLLARY 2.3. Let $A \in \text{int}(\Pi_n)$. Then
$A \ast A - (A \ast I)(A^{-1} \ast A + I)^{-1}(A \ast I)$

is in $\Pi_n$.

**Proof.** Let

$M = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}$, $N = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$, and $P = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}$.

Then $M \ast N + P \in \Pi_{2n}$, and the result follows as in the previous corollary.

In fact, even more is known concerning Corollary 2.3. In [9, Corollary 4.3, p. 236], Styan shows that $A \ast A - 2(A \ast I)(A^{-1} \ast A + I)^{-1}(A \ast I) \in \Pi_n$ using a technique based on probabilistic methods.

We also would like to point out that Theorem 2 is an analogue for the Schur product of Theorem 5 of [2]. There it is shown that if $A, C \in \Pi_n$, and if $B, D$ are chosen so that $N(A) \subseteq N(B^\ast), N(C) \subseteq N(D^\ast)$, then

$B^\ast A^\ast B + D^\ast C^\ast D - (B + D)^\ast (A + C)^\ast (B + D) \in \Pi_n$.

From Corollary 2.2, if $A, B \in \text{int}(\Pi_n)$, then it follows that $A \ast B - (A^{-1} \ast B^{-1})^{-1} \in \Pi_n$. There is an analogue of this result for matrix addition, i.e. $A + B - (A^{-1} + B^{-1})^{-1} \in \text{int}(\Pi_n)$. This is a consequence of the previously mentioned result of Carlson, Haynsworth and Markham [2]; we offer a simple proof of this fact.

Let

$M = \begin{pmatrix} A & \frac{1}{2} I \\ \frac{1}{2} I & A^{-1} \end{pmatrix}$ and $N = \begin{pmatrix} B & \frac{1}{2} I \\ \frac{1}{2} I & B^{-1} \end{pmatrix}$.

By Theorem 1.2, both $M$ and $N$ belong to $\text{int}(\Pi_{2n})$. Now

$M + N = \begin{pmatrix} A + B & I \\ I & A^{-1} + B^{-1} \end{pmatrix} \in \text{int}(\Pi_{2n})$.

Apply Theorem 1.2 again. Then $M + N[A^{-1} + B^{-1}] \in \text{int}(\Pi_n)$. But $M + N[A^{-1} + B^{-1}] = A + B - (A^{-1} + B^{-1})^{-1}$. □

**III. M-matrices.** Suppose $A$ is a square matrix over the real field. Let $Z^n$ denote the class of $n \times n$ matrices whose off-diagonal entries are nonpositive. Assume $A \in Z^n$. $A$ is called an $M$-matrix, see [6], if and only if $A$ is invertible and $A^{-1}$ is a nonnegative matrix (each entry is nonnegative). Let

$(3.1) G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

where $A$ and $D$ are square matrices of order $k$ and $n - k$, respectively.

If $G$ is an $M$-matrix, then it is well known that $A$ and $D$ are $M$-matrices. Fan [5] proved that if $D$ has order 1, then $G \mid D$ is an $M$-matrix. Crabtree [3, Lemma 1] extended this result to $D$ of arbitrary order. Watford [10], in turn, proved this result for generalized $M$-matrices with respect to a cone.
These results are useful in obtaining an analogue of Albert's Theorem 1.2 for $M$-matrices.

**Theorem 3.** Suppose $G$ is an $n \times n$ matrix partitioned as in (3.1), and $G$ is in $Z_n$. Then $G$ is an $M$-matrix if and only if $A$, $D$, $G|A$, and $G|D$ are $M$-matrices.

**Proof.** If $G$ is an $M$-matrix, then $A$, $D$, $G|A$, and $G|D$ are $M$-matrices by the comments preceding Theorem 3.

Now suppose $A$, $D$, $G|A$, $G|D$ are $M$-matrices. Let

$$
\tilde{G} = \begin{bmatrix}
(G|D)^{-1} & -A^{-1}B(G|A)^{-1} \\
-D^{-1}C(G|D)^{-1} & (G|A)^{-1}
\end{bmatrix}.
$$

It is easy to verify $G \cdot \tilde{G} = I$, so $G^{-1}$ exists. Further, $G^{-1}$ is nonnegative since each of $A^{-1}$, $D^{-1}$, $(G|A)^{-1}$, and $(G|D)^{-1}$ is nonnegative, and $B$ and $C$ are nonpositive. Thus $G$ is an $M$-matrix. \(\Box\)

Theorem 3 offers a practical procedure for determining if a given matrix is an $M$-matrix.

Now we will take a closer look at Albert's theorem. First, we need some additional notation. If $\alpha$ and $\beta$ are strictly increasing sequences on $\{1, 2, \ldots, n\}$ of the same length, then $M(\alpha|\beta)$ will denote the minor of $M$ with rows indexed by $\alpha$ and columns indexed by $\beta$. If $\alpha = \beta$, then we write $M(\alpha)$. If $M$ is partitioned as in (1.1), where $A$ is nonsingular of order $k$, then $M|A = (e_{ij})$, $i, j = k + 1, \ldots, n$, with

$$
e_{ij} = \frac{M(1, 2, \ldots, k, i\{1, 2, \ldots, k, j\})}{M(1, 2, \ldots, k)} = \frac{M(1, 2, \ldots, k, i\{1, 2, \ldots, k, j\})}{\det(A)};$$

see [4].

If $M$ is hermitian, then $M$ is positive definite if and only if the leading principal minors of $M$ are positive. Hence we can rephrase Albert's theorem for this case.

**Theorem 4.** Suppose $M$ is hermitian, and is partitioned symmetrically in (1.1). Then $M \in \text{int}(\Pi_q)$ if and only if $A \in \text{int}(\Pi_k)$ and $M|A \in \text{int}(\Pi_{n-k})$.

**Proof.** It is well known that if $M \in \text{int}(\Pi_q)$, then $A$ and $M|A$ are positive definite.

Conversely, we need only show that the leading principal minors of $M$ are positive. Consider an arbitrary minor, say $M(1, \ldots, i_p)$. If $i_p < k$, this minor is positive since it is a principal minor of $A$. Assume $i_p > k$. Then, using an identity of Sylvester [7, p. 101], we have

$$M|A(k + 1, \ldots, i_p) = (\det(A)^* \cdot M(1, \ldots, k, k + 1, \ldots, i_p)^{-1}.$$ 

The result now follows. \(\Box\)

If $M \in Z$, then $M$ is an $M$-matrix if and only if the leading principal
minors of $M$ are positive. Thus, Theorem 3 could also be restated in the form of Theorem 4.

**Definition [6].** Suppose $M$ is an $n \times n$ matrix. Then $M$ belongs to class $P$ if and only if all principal minors of $M$ are positive.

We can generalize Albert's theorem to class $P$ in the following manner.

**Theorem 5.** Let $M$ be partitioned as in (1.1), where the submatrix $A$ has order 1. Then

\[ M \in P \text{ if and only if } A \in P, \quad M|A \in P, \quad \text{and } D \in P. \]

We omit the proof since the techniques are similar to those of Theorem 4.

Observe the following concerning Theorem 5. On the one hand, to see if $M \in P$, there are $2^n - 1$ principal minors to check. Applying the above result, we obtain a number and two matrices of order $n - 1$ to check the principal minors. Using this equivalence iteratively (to the right-hand side of (3.3)), there are $1 + 2 + \cdots + 2^{n-1}$ numbers which must be verified to be positive. But $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$ for $n$ a positive integer, so, in fact, the same number of elements must be verified. The obvious advantage of the right-hand side of (3.3) lies in the reduction of the order of the matrices at each iteration.

It is possible to reduce the number of minors checked? For example, if $M$ has leading positive principal minors, then $M$ does not necessarily belong to class $P$. A simple example to illustrate is $M = \begin{bmatrix} 1 & * & 0 \\ -1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$.

Does there exist an analogue to Theorem 3 for class $P$ when $M$ is partitioned as in (1.1), with $A$ of order $k$? If $M$ has order 2 or 3, the result holds. For larger orders, it need not hold. Consider

\[ M = \begin{bmatrix} 1 & 1 & \hline & 1 & 0 \\ \hline 1 & 2 & \hline & 0 & 1 \\ \hline \frac{1}{2} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} & \hline & 1 & 1 \\ \hline \end{bmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]

Here $A$, $M|A$, $D$, and $M|D$ are all in class $P$, but $M(13)$ is zero.

We conclude with the following query. Suppose $M$ is an $n \times n$ matrix. What is the minimal number of principal minors of $M$ that must be positive in order that $M$ belong to class $P$? Is it necessary to verify that all $2^n - 1$ principal minors, or related minors, are positive?

**References**


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