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ON THE COHOMOLOGY GROUPS OF A MANIFOLD WITH A NONINTEGRABLE SUBBUNDLE

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ABSTRACT. We define the cohomology groups $H^u(M, \mathfrak{D}^{s,t})$ of a manifold M with a nonintegrable subbundle E, and we give the condition for the existence of a bundle-like metric with respect to E.

1. **Introduction**. N. Abe [1], J. L. Heitsch [3] and I. Vaisman [6] studied some cohomology groups of a manifold with a foliation ("integrable case"). In this note, we generalize their results to the nonintegrable case, that is, we define the cohomology groups $H^u(M, \mathfrak{D}^{s,t})$ of a manifold M with a nonintegrable subbundle E. In the case that M is a riemannian manifold, by its cohomology groups, we give the condition for the existence of a bundle-like metric with respect to E.

2. **Preliminaries**. We shall be in C^∞-category. Let M be an n-dimensional paracompact manifold with tangent bundle TM. Let E be a subbundle of TM with the constant fibre dimension $n - p$ $(0 < p < n)$. We assume that E is not integrable. $\Gamma(\cdot)$ denotes the functor associating to a bundle its vector space of sections, and $[\ ,\ ]$ the bracket operator on $\Gamma(TM)$. Let $C(E)$ be the "Cauchy characteristic subbundle" of E, i.e. the fibre $C_x(E)$ over $x \in M$ of $C(E)$ consists of $X_x \in E_x$ (= the fibre over x of E) such that $[X, Y]_x \in E_x$ for any $Y_x \in E_x$, $X, Y \in \Gamma(E)$, and, for all $x \in M$, dim $C_x(E)$ is assumed to be constant. Then $C(E)$ is an integrable subbundle of E (naturally, of TM). We assume that the fibre dimension of $C(E)$ is $n - q$ $(0 < p < q < n)$. We set

$$Q = TM/E, \qquad E' = E/C(E), \tag{1}$$

and, by a suitable riemannian metric on TM, we have isomorphisms

$$\begin{gathered} \Gamma(TM) = \Gamma(Q) \oplus \Gamma(E), \qquad \Gamma(TM) = \Gamma(TM/C(E)) \oplus \Gamma(C(E)), \\ \Gamma(TM) = \Gamma(Q) \oplus \Gamma(E') \oplus \Gamma(C(E)). \end{gathered} \tag{2}$$

3. **(s, t, u)-forms and cohomology groups $H^u(M, \mathfrak{D}^{s,t})$**. Let A^r be the space of all r-forms on M and d the exterior derivative.

DEFINITION. *An r-form $\omega \in A^r$ is a (s, t, u)-form, if*

(i) $s + t + u = r$, *and*

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(ii) $\omega(X_1, \ldots, X_r) = 0$ *except for s arguments X_i's in* $\Gamma(Q)$, *t arguments X_i's in* $\Gamma(E')$ *and u arguments X_i's in* $\Gamma(C(E))$.

Let $A^{s,t,u}$ be the space of all (s, t, u)-forms on M, and we have a decomposition

$$A^r = \sum_{s+t+u=r} A^{s,t,u}.$$

By the definition of $C(E)$, we have

(3) $[\Gamma(C(E)), \Gamma(E)] \subset \Gamma(E), \qquad [\Gamma(C(E)), \Gamma(C(E))] \subset \Gamma(C(E))$.

From this, we have that the partial derivative

$$\hat{\partial}\colon A^{s,t,u} \to A^{s,t,u+1}$$

induced by the exterior derivative satisfies $(\hat{\partial})^2 = 0$. Let $\mathcal{A}^{s,t,u}$ be the sheaf of germs of (s, t, u)-forms. Then each $\mathcal{A}^{s,t,u}$ is a fine sheaf. Let $\mathcal{D}^{s,t}$ be the sheaf defined by $\mathcal{D}^{s,t} = \mathcal{A}^{s,t,0} \cap \ker(\hat{\partial})$.

REMARK. $\mathcal{D}^{0,0}$ denotes the sheaf of germs of functions which are constants on the leaves of $C(E)$.

THEOREM 1. *There exists a fine resolution of the sheaf*

$$\mathcal{D}^{s,t}\colon 0 \to \mathcal{D}^{s,t} \xrightarrow{i} \mathcal{A}^{s,t,0} \xrightarrow{\hat{\partial}} \mathcal{A}^{s,t,1} \xrightarrow{\hat{\partial}} \mathcal{A}^{s,t,2} \xrightarrow{\hat{\partial}} \mathcal{A}^{s,t,3} \xrightarrow{\hat{\partial}} \cdots,$$

where i denotes the natural inclusion.

PROOF. Since we can obtain the Poincaré lemma for the operator $\hat{\partial}$ and the (s, t, u)-form on the open unit ball in euclidean n-space (cf. [6]), we easily prove the assertion of the theorem.

Let $H^u(M, \mathcal{D}^{s,t})$ be the cohomology groups of M with coefficients in the sheaf $\mathcal{D}^{s,t}$. Then we have

THEOREM 2. *There exist isomorphisms*

$$H^0(M, \mathcal{D}^{s,t}) \cong A^{s,t,0} \cap \ker(\hat{\partial}),$$

$$H^u(M, \mathcal{D}^{s,t}) \cong A^{s,t,u} \cap \ker(\hat{\partial})/\hat{\partial}A^{s,t,u-1} \quad \textit{for } u \geqslant 1.$$

COROLLARY 3. $H^u(M, \mathcal{D}^{s,t}) \cong \{0\}$ *for* $s > p$ *or* $t > q - p$ *or* $u > n - q$.

4. **Generalized Bott connection and cohomology groups** $H^u(M, \mathcal{D}^{s,t}(Q))$. Let $\pi\colon TM \to Q = TM/E$ be the canonical projection. We define a map

$$\hat{\nabla}\colon \Gamma(C(E)) \times \Gamma(Q) \to \Gamma(Q)$$

by

$$\hat{\nabla}_X(S) = \pi_*([X, \tilde{S}]) \tag{4}$$

for $\forall X \in \Gamma(C(E))$, $\forall S \in \Gamma(Q)$ and $\tilde{S} \in \Gamma(TM)$ such that $\pi_*(\tilde{S}) = S$. From (3), this is well defined. Let ∇' be any connection on Q. For $X \in \Gamma(TM)$, from (2), we can write $X = X_1 + X_2$, $X_1 \in \Gamma(C(E))$, $X_2 \in \Gamma(TM/C(E))$. Thus we define a map

$$\nabla\colon \Gamma(TM) \times \Gamma(Q) \to \Gamma(Q)$$

by

$$\nabla_X(S) = \hat{\nabla}_{X_1}(S) + \nabla'_{X_2}(S). \tag{5}$$

Then ∇ is a connection on Q, and is called a generalized Bott connection (cf. [2], [4]).

Let $A^{s,t,u}(Q)$ be the space of all Q-valued (s, t, u)-forms on M and $\mathcal{A}^{s,t,u}(Q)$ the corresponding sheaf. $\hat{\partial}$ operating on $A^{s,t,u}(Q)$ is given by ∇. Then, as above, we have a fine resolution of the sheaf $\mathcal{D}^{s,t}(Q) = \mathcal{A}^{s,t,0}(Q) \cap \ker(\hat{\partial})$. Thus we have

THEOREM 4. *There exist isomorphisms*:

$$H^0(M, \mathcal{D}^{s,t}(Q)) \cong A^{s,t,0}(Q) \cap \ker(\hat{\partial}),$$

$$H^u(M, \mathcal{D}^{s,t}(Q)) \cong A^{s,t,u}(Q) \cap \ker(\hat{\partial})/\hat{\partial}A^{s,t,u-1}(Q) \quad \textit{for } u \geqslant 1.$$

REMARK. If E is integrable, the same results are given by J. L. Heitsch [3].

5. **Bundle-like metric with respect to** E. Let M be an n-dimensional riemannian manifold with the metric g, and $E^{\perp}$ the orthogonal complement of E in TM. Let $\hat{\nabla}$: $\Gamma(C(E)) \times \Gamma(E^{\perp}) \to \Gamma(E^{\perp})$ be a map defined by $\hat{\nabla}_X(S) = \pi_*([X, S])$ (π: $TM \to E^{\perp}$ the canonical projection), we define a connection ∇ on $E^{\perp}$ as in (5).

DEFINITION. *The riemannian metric g is a bundle-like metric with respect to E, if $(\nabla_X g)(S_1, S_2) = 0$ for $\forall X \in \Gamma(C(E))$, $\forall S_1$, $\forall S_2 \in \Gamma(E^{\perp})$.*

In the following, we assume that the fibre dimension of E is $n - 1$. Let $\{e_A\}$ be an orthonormal frame such that $e_1 \in \Gamma(E^{\perp})$ and $e_a \in \Gamma(E)$, and $\{\omega^A\}$ its dual ($1 \leqslant A \leqslant n$, $2 \leqslant a \leqslant n$). We assume that ω^1 is a global form (if necessary, we assume that E is transversally orientable).

LEMMA 5. *g is a bundle-like metric with respect to E if and only if $\nabla_X(e_1) = 0$ for $\forall X \in \Gamma(C(E))$.*

PROOF. For $\forall S_1 = \xi \cdot e_1$, $\forall S_2 = \eta \cdot e_1 \in \Gamma(E^{\perp})$ (ξ, η: functions),

$$\begin{aligned}(\nabla_X g)(S_1, S_2) &= X(g(S_1, S_2)) - g(\nabla_X(S_1), S_2) - g(S_1, \nabla_X(S_2))\\ &= X(\xi \cdot \eta) - \eta \cdot X(\xi) - (\xi \cdot \eta) g(\nabla_X(e_1), e_1)\\ &\quad - \xi \cdot X(\eta) - (\xi \cdot \eta) g(e_1, \nabla_X(e_1))\\ &= -2(\xi \cdot \eta) g(e_1, \nabla_X(e_1)).\end{aligned}$$

Thus we have the assertion of the lemma.

By the above metric g, we have an isomorphism $\Gamma(Q) \cong \Gamma(E^{\perp})$, and we can identify the connections ∇ on Q and on $E^{\perp}$.

LEMMA 6. $\hat{\partial}\omega^1 = 0$ *if and only if* $\nabla_X(e_1) = 0$ *for* $\forall X \in \Gamma(C(E))$.

PROOF. For $\forall S = \xi \cdot e_1$ (ξ: function),

$$\begin{aligned}\hat{\partial}\omega^1(S, X) &= -X(\omega^1(S)) - \omega^1([S, X]) = -X(\xi) + \omega^1(\nabla_X(S)) \\ &= -X(\xi) + X(\xi) + \xi \cdot \omega^1(\nabla_X(e_1)) \\ &= \xi \cdot \omega^1(\nabla_X(e_1)).\end{aligned}$$

Thus we have the assertion of the lemma.

From the above lemmas, we have

THEOREM 7. *Let M be an n-dimensional riemannian manifold with the metric g and E a nonintegrable, transversally orientable subbundle of TM of fibre dimension $n-1$. If $H^0(M, \mathfrak{D}^{1,0}) \cong A^{1,0,0}$, then g is a bundle-like metric with respect to E. Conversely, if g is a bundle-like metric with respect to E, then $H^0(M, \mathfrak{D}^{1,0}) \neq \{0\}$.*

PROOF. A nonzero 1-form ω^1 is a $(1, 0, 0)$-form on M. If $H^0(M, \mathfrak{D}^{1,0}) \cong A^{1,0,0}$, then $A^{1,0,0} \cong A^{1,0,0} \cap \ker(\hat{\partial})$ and we have $\hat{\partial}\omega^1 = 0$. By Lemmas 5 and 6, g is a bundle-like metric with respect to E. Conversely, if g is a bundle-like metric with respect to E, by Lemmas 5 and 6, we have $\hat{\partial}\omega^1 = 0$ and ω^1 is a nonzero $(1, 0, 0)$-form on M. Thus we have $H^0(M, \mathfrak{D}^{1,0}) \neq \{0\}$.

REMARK. In the case that E is integrable, M is compact and g is a bundle-like metric, then $H^1(M, R) \neq \{0\}$ (cf. R. Sacksteder [5]).

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