ON THE COHOMOLOGY GROUPS OF A MANIFOLD WITH A NONINTEGRABLE SUBBUNDLE

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Abstract. We define the cohomology groups $H^*(M, \Omega^{n,*})$ of a manifold $M$ with a nonintegrable subbundle $E$, and we give the condition for the existence of a bundle-like metric with respect to $E$.

1. Introduction. N. Abe [1], J. L. Heitsch [3] and I. Vaisman [6] studied some cohomology groups of a manifold with a foliation ("integrable case"). In this note, we generalize their results to the nonintegrable case, that is, we define the cohomology groups $H^*(M, \Omega^{n,*})$ of a manifold $M$ with a nonintegrable subbundle $E$. In the case that $M$ is a riemannian manifold, by its cohomology groups, we give the condition for the existence of a bundle-like metric with respect to $E$.

2. Preliminaries. We shall be in $C^\infty$-category. Let $M$ be an $n$-dimensional paracompact manifold with tangent bundle $TM$. Let $E$ be a subbundle of $TM$ with the constant fibre dimension $n - p$ ($0 < p < n$). We assume that $E$ is not integrable. $\Gamma(\cdot)$ denotes the functor associating to a bundle its vector space of sections, and $[ , ]$ the bracket operator on $\Gamma(TM)$. Let $C(E)$ be the "Cauchy characteristic subbundle" of $E$, i.e. the fibre $C_x(E)$ over $x \in M$ of $C(E)$ consists of $X_x \in E_x$ (= the fibre over $x$ of $E$) such that $[X, Y]_x \in E_x$ for any $Y_x \in E_x$, $X, Y \in \Gamma(E)$, and, for all $x \in M$, dim $C_x(E)$ is assumed to be constant. Then $C(E)$ is an integrable subbundle of $E$ (naturally, of $TM$). We assume that the fibre dimension of $C(E)$ is $n - q$ ($0 < p < q < n$). We set

(1) $Q = TM/E$, $E' = E/C(E)$,

and, by a suitable riemannian metric on $TM$, we have isomorphisms

(2) $\Gamma(TM) = \Gamma(Q) \oplus \Gamma(E)$, $\Gamma(TM) = \Gamma(TM/C(E)) \oplus \Gamma(C(E))$.

3. $(s, t, u)$-forms and cohomology groups $H^*(M, \Omega^{n,*})$. Let $A'$ be the space of all $r$-forms on $M$ and $d$ the exterior derivative.

Definition. An $r$-form $\omega \in A'$ is a $(s, t, u)$-form, if

(i) $s + t + u = r$, and

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Let $A^{s,t,u}$ be the space of all $(s, t, u)$-forms on $M$, and we have a decomposition

$$A^r = \sum_{s+t+u=r} A^{s,t,u}.$$  

By the definition of $C(E)$, we have

$$[\Gamma(C(E)), \Gamma(E)] \subset \Gamma(E), \quad [\Gamma(C(E)), \Gamma(C(E))] \subset \Gamma(C(E)).$$

From this, we have that the partial derivative

$$\delta : A^{s,t,u} \to A^{s,t,u+1}$$

induced by the exterior derivative satisfies $(\delta)^2 = 0$. Let $\mathcal{A}^{s,t,u}$ be the sheaf of germs of $(s, t, u)$-forms. Then each $\mathcal{A}^{s,t,u}$ is a fine sheaf. Let $\mathcal{O}^{s,t}$ be the sheaf defined by $\mathcal{O}^{s,t} = \mathcal{A}^{s,t,0} \cap \ker(\delta)$.

**Remark.** $\mathcal{O}^{0,0}$ denotes the sheaf of germs of functions which are constants on the leaves of $C(E)$.

**Theorem 1.** There exists a fine resolution of the sheaf

$$0 \to \mathcal{A}^{s,t,u} \to \mathcal{A}^{s,t,u+1} \to \mathcal{A}^{s,t,u+2} \to \ldots,$$

where $i$ denotes the natural inclusion.

**Proof.** Since we can obtain the Poincaré lemma for the operator $\delta$ and the $(s, t, u)$-form on the open unit ball in euclidean $n$-space (cf. [6]), we easily prove the assertion of the theorem.

Let $H^u(M, \mathcal{O}^{s,t})$ be the cohomology groups of $M$ with coefficients in the sheaf $\mathcal{O}^{s,t}$. Then we have

**Theorem 2.** There exist isomorphisms

$$H^0(M, \mathcal{O}^{s,t}) \cong A^{s,t,0} \cap \ker(\delta),$$

$$H^u(M, \mathcal{O}^{s,t}) \cong A^{s,t,u} \cap \ker(\delta) / \delta A^{s,t,u-1} \quad \text{for } u \geq 1.$$  

**Corollary 3.** $H^u(M, \mathcal{O}^{s,t}) = \{0\}$ for $s > p$ or $t > q - p$ or $u > n - q$.

4. Generalized Bott connection and cohomology groups $H^u(M, \mathcal{O}^{s,t}(Q))$. Let $\pi : TM \to Q = TM/E$ be the canonical projection. We define a map

$$\hat{\nabla} : \Gamma(C(E)) \times \Gamma(Q) \to \Gamma(Q)$$

by

$$\hat{\nabla}_X(S) = \pi_*([X, \bar{S}])$$

for $\forall X \in \Gamma(C(E)), \forall S \in \Gamma(Q)$ and $\bar{S} \in \Gamma(TM)$ such that $\pi_*\bar{S} = S$.

From (3), this is well defined. Let $\nabla'$ be any connection on $Q$. For $X \in \Gamma(TM)$, from (2), we can write $X = X_1 + X_2$, $X_1 \in \Gamma(C(E))$, $X_2 \in \Gamma(TM/C(E))$. Thus we define a map

$$\nabla : \Gamma(TM) \times \Gamma(Q) \to \Gamma(Q)$$
by
\[ (5) \quad \nabla_X(S) = \hat{\nabla}_{X_1}(S) + \nabla_{X_2}(S). \]

Then \( \nabla \) is a connection on \( Q \), and is called a generalized Bott connection (cf. [2], [4]).

Let \( A^{s,t,u}(Q) \) be the space of all \( Q \)-valued \((s, t, u)\)-forms on \( M \) and \( \hat{\omega}^{s,t,u}(Q) \) the corresponding sheaf. \( \hat{\omega} \) operating on \( A^{s,t,u}(Q) \) is given by \( \nabla \). Then, as above, we have a fine resolution of the sheaf \( \hat{\omega}^{s,t}(Q) = \hat{\omega}^{s,t,u}(Q) \cap \ker(\hat{\omega}) \). Thus we have

**Theorem 4.** There exist isomorphisms:

\[ H^0(M, \hat{\omega}^{s,t}(Q)) \cong A^{s,t,u}(Q) \cap \ker(\hat{\omega}), \]
\[ H^u(M, \hat{\omega}^{s,t}(Q)) \cong A^{s,t,u}(Q) \cap \ker(\hat{\omega}) / \hat{\omega} A^{s,t,u-1}(Q) \quad \text{for } u \geq 1. \]

**Remark.** If \( E \) is integrable, the same results are given by J. L. Heitsch [3].

5. **Bundle-like metric with respect to \( E \).** Let \( M \) be an \( n \)-dimensional riemannian manifold with the metric \( g \), and \( E^\perp \) the orthogonal complement of \( E \) in \( TM \). Let \( \hat{\nabla} : \Gamma(C(E)) \times \Gamma(E^\perp) \to \Gamma(E^\perp) \) be a map defined by \( \hat{\nabla}_X(S) = \pi_*[X, S] \) (\( \pi : TM \to E^\perp \) the canonical projection), we define a connection \( \nabla \) on \( E^\perp \) as in (5).

**Definition.** The riemannian metric \( g \) is a bundle-like metric with respect to \( E \), if \( (\hat{\nabla}_Xg)(S_1, S_2) = 0 \) for \( X \in \Gamma(C(E)), S_1, S_2 \in \Gamma(E^\perp) \).

In the following, we assume that the fibre dimension of \( E \) is \( n - 1 \). Let \( \{e^A\} \) be an orthonormal frame such that \( e_1 \in \Gamma(E^\perp) \) and \( e_a \in \Gamma(E) \), and \( \{\omega^a\} \) its dual (\( 1 < A < n, 2 < a < n \)). We assume that \( \omega^1 \) is a global form (if necessary, we assume that \( E \) is transversally orientable).

**Lemma 5.** \( g \) is a bundle-like metric with respect to \( E \) if and only if \( \nabla_X(e_1) = 0 \) for \( X \in \Gamma(C(E)) \).

**Proof.** For \( \forall S_1 = \xi \cdot e_1, \forall S_2 = \eta \cdot e_1 \in \Gamma(E^\perp) \) (\( \xi, \eta \): functions),
\[
(\nabla_Xg)(S_1, S_2) = X(g(S_1, S_2)) - g(\nabla_X(S_1), S_2) - g(S_1, \nabla_X(S_2))
\]
\[= X(\xi \cdot \eta) - \eta \cdot X(\xi) - (\xi \cdot \eta)g(\nabla_X(e_1), e_1)
\]
\[= -2(\xi \cdot \eta)g(e_1, \nabla_X(e_1)).\]

Thus we have the assertion of the lemma.

By the above metric \( g \), we have an isomorphism \( \Gamma(Q) \cong \Gamma(E^\perp) \), and we can identify the connections \( \nabla \) on \( Q \) and on \( E^\perp \).

**Lemma 6.** \( \hat{\omega} \omega^1 = 0 \) if and only if \( \nabla_X(e_1) = 0 \) for \( X \in \Gamma(C(E)) \).

**Proof.** For \( \forall S = \xi \cdot e_1 \) (\( \xi \): function),
\[ \hat{\omega}^1(S, X) = -X(\omega^1(S)) - \omega^1([S, X]) = -X(\xi) + \omega^1(\nabla_X(S)) \]
\[ = -X(\xi) + X(\xi) + \xi \cdot \omega^1(\nabla_X(e_i)) \]
\[ = \xi \cdot \omega^1(\nabla_X(e_i)). \]

Thus we have the assertion of the lemma.

From the above lemmas, we have

**Theorem 7.** Let \( M \) be an \( n \)-dimensional riemannian manifold with the metric \( g \) and \( E \) a nonintegrable, transversally orientable subbundle of \( TM \) of fibre dimension \( n - 1 \). If \( H^0(M, \otimes^{1,0}) \cong A^{1,0} \), then \( g \) is a bundle-like metric with respect to \( E \). Conversely, if \( g \) is a bundle-like metric with respect to \( E \), then \( H^0(M, \otimes^{1,0}) \neq \{0\} \).

**Proof.** A nonzero 1-form \( \omega^1 \) is a \((1, 0, 0)\)-form on \( M \). If \( H^0(M, \otimes^{1,0}) \cong A^{1,0} \), then \( A^{1,0,0} = A^{1,0,0} \cap \ker(\hat{\omega}) \) and we have \( \hat{\omega}^1 = 0 \). By Lemmas 5 and 6, \( g \) is a bundle-like metric with respect to \( E \). Conversely, if \( g \) is a bundle-like metric with respect to \( E \), by Lemmas 5 and 6, we have \( \hat{\omega}^1 = 0 \) and \( \omega^1 \) is a nonzero \((1, 0, 0)\)-form on \( M \). Thus we have \( H^0(M, \otimes^{1,0}) \neq \{0\} \).

**Remark.** In the case that \( E \) is integrable, \( M \) is compact and \( g \) is a bundle-like metric, then \( H^1(M, R) \neq \{0\} \) (cf. R. Sacksteder [5]).

**References**

1. N. Abe, On cohomology and characteristic classes of a foliated manifold (preprint).
4. H. Kitahara and S. Yorozu, A generalized Godbillon-Vey invariant for a subbundle which is not integrable (preprint).

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