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# ANALYTIC TOEPLITZ OPERATORS WITH AUTOMORPHIC SYMBOL. II

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ABSTRACT. For $\phi$ in $H^\infty$, let $T_\phi$ be the analytic Toeplitz operator with symbol $\phi$ and let $\{T_\phi\}'$ be the commutant of $T_\phi$. Two infinite Blaschke products $\phi$ and $\psi$ are exhibited such that $\{T_\phi\}' \cap \{T_\psi\}'$ is not equal to $\{T_\theta\}'$ for any inner function $\theta$. Also, two questions on reducing subspaces of analytic Toeplitz operators are answered.

1. **Introduction.** For $\phi$ in $H^\infty$ of the unit disk, the analytic Toeplitz operator $T_\phi$ on $H^2$ is defined by $T_\phi(f) = \phi f$. The commutant of $T_\phi$ is the set of operators $S$ on $H^2$ such that $ST_\phi = T_\phi S$ and is denoted $\{T_\phi\}'$. In [4], Deddens and Wong ask the following question.

*Question* 1. Suppose $\{\chi_\alpha: \alpha \text{ in } \mathcal{Q}\}$ is a family of inner functions. Is $\bigcap_{\alpha \in \mathcal{Q}}\{T_{\chi_\alpha}\}'$ equal to $\{T_\theta\}'$ where $\theta$ is some inner function of which each $\chi_\alpha$ is a function?

James Thomson has shown that if one of the $\chi_\alpha$ is a finite Blaschke product, then the answer to Question 1 is affirmative [13]. In this paper it is shown that Thomson's result is sharp. In fact, we produce two infinite Blaschke products $\phi$ and $\psi$ such that $\{T_\phi\}' \cap \{T_\psi\}'$ does not equal $\{T_\theta\}'$ for any inner function $\theta$.

The second author has raised the following two questions on reducing subspaces of analytic Toeplitz operators [3].

*Question* 2. If $\{\chi_\alpha: \alpha \text{ in } \mathcal{Q}\}$ is a collection of inner functions, if $\chi$ is the greatest common divisor of the $\chi_\alpha$, and if $\mathfrak{M}$ is a closed subspace of $H^2$ which reduces each $T_{\chi_\alpha}$, must $\mathfrak{M}$ reduce $T_\chi$?

*Question* 3. If $\phi$ in $H^\infty$ has inner-outer factorization $\phi = \chi F$ and if $\mathfrak{M}$ is a closed subspace of $H^2$ which reduces $T_\phi$, must $\mathfrak{M}$ reduce $T_\chi$ and $T_F$?

It is shown that the status of Question 2 is the same as that of Question 1: the answer is affirmative if one of the $\chi_\alpha$ is a finite Blaschke product and there is a pair of infinite Blaschke products for which the answer is negative. The answer to Question 3 is also shown to be negative.

The counterexample for Question 1 makes use of the theory of bundle shifts developed by the first author and R. G. Douglas [2] and applied

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previously to analytic Toeplitz operators by the first author [1]. It also makes use of a result of Rudin [9] and Stout [12] on inner generators of the space of rational functions on an annulus. The proof of the affirmative part of the answer to Question 2 uses the aforementioned result of Thomson [13] and the counterexamples for Questions 2 and 3 make use of a composition operator as in [1]. Here, the composition operator is preceded by a multiplication operator that makes the product unitary and the reducing subspace is the range of a projection in the $W^*$-algebra generated by this unitary. This technique of perturbing a composition operator to make it unitary was suggested to the first author several years ago by R. G. Douglas.

2. **Automorphic inner functions**. Let $D$ denote the unit disk $\{z: |z| < 1\}$, let $R$ denote the annulus $\{z: \frac{1}{2} < |z| < 1\}$, and let $\pi$ be the holomorphic universal covering map from $D$ onto $R$ as defined in [11] and [1] by

$$\pi(z) = \exp\left( \frac{i}{\pi} \log \frac{1}{2} \operatorname{Log} \frac{1+z}{1-z} + \frac{1}{2} \log \frac{1}{2} \right)$$

where Log is the principal branch of the logarithm. It is shown in [1] that $T_\pi$ is a pure subnormal operator with spectrum contained in the closure of $R$ and normal spectrum contained in the boundary of $R$ and thus, by [2, Theorem 11], there is a vector bundle $E$ over $R$ such that the bundle shift $S_E$ is unitarily equivalent to $T_\pi$. The bundle shift $S_E$ is multiplication by $z$ on the space $H_E^2(R)$ of $H^2$ cross-sections of the bundle $E$. Let $A(R)$ be the space of continuous functions on the closure of $R$ that can be approximated uniformly by rational functions with poles off the closure of $R$ and for $\phi$ in $A(R)$ let $T_\phi^E$ be the operator on $H_E^2(R)$ defined by $T_\phi^E(f) = \phi f$. It is easily verified that $\phi(T_\pi) = T_{\phi \circ \pi}$ and $\phi(S_E) = T_\phi^E$ for all $\phi$ in $A(R)$. This establishes the following lemma.

LEMMA 2.1. *There is a unitary operator $V$ from $H^2(D)$ onto $H_E^2(R)$ such that $VT_{\phi \circ \pi} = T_\phi^E V$ for all $\phi$ in $A(R)$.*

A generating set for $A(R)$ is a subset $G$ of $A(R)$ such that the smallest uniformly closed subalgebra of $A(R)$ containing $G$ is all of $A(R)$. The space $H^\infty(R)$ is the Banach algebra of all bounded analytic functions on $R$. For a set of operators $\mathfrak{S}$, the second commutant of $\mathfrak{S}$ is the commutant of the commutant of $\mathfrak{S}$ and is denoted $\mathfrak{S}''$.

LEMMA 2.2. *If $G$ is a generating set for $A(R)$, then the Banach algebra $\{T_{\phi \circ \pi}: \phi \text{ in } G\}''$ is isomorphic to $H^\infty(R)$.*

PROOF. By Lemma 2.1, the algebra $\{T_{\phi \circ \pi}: \phi \text{ in } G\}''$ is unitarily equivalent to the algebra $\{T_\phi^E: \phi \text{ in } G\}''$. Since $G$ is a generating set for $A(R)$, the latter algebra is equal to the second commutant of the bundle shift $S_E$. The result now follows from [2, Theorem 4].

The function in $A(R)$ is said to be inner if it is unimodular on the boundary of $R$. In the following lemma and elsewhere in this paper, a function which is a Blaschke product times a scalar of unit modulus shall be referred to as a Blaschke product.

LEMMA 2.3. *If $\phi$ is a nonconstant inner function in $A(R)$, then $\phi \circ \pi$ is an infinite Blaschke product.*

PROOF. The covering map $\pi$ is continuous on the set $\{z: |z| \leqslant 1, z \neq 1, z \neq -1\}$ and maps the sets $\{z: |z| = 1, \operatorname{Im} z > 0\}$ and $\{z: |z| = 1, \operatorname{Im} z < 0\}$ onto the outer and inner boundaries of $R$ respectively. It follows that $\phi \circ \pi$ is an inner function of the form

$$\phi(\pi(z)) = \lambda B(z)\exp\left(-a\frac{1+z}{1-z}\right)\exp\left(-b\frac{-1+z}{-1-z}\right)$$

where $|\lambda| = 1$, $B$ is a Blaschke product, and $a$ and $b$ are nonnegative real numbers. If $a$ is not equal to zero, then

$$0 = \lim_{x \uparrow 1} \phi(\pi(x)) = \lim_{t \to \infty} \phi(e^{it}/\sqrt{2}).$$

(Here one uses the fact that $\pi$ maps the interval $(-1, 1)$ around the circle $\{|z| = 1/\sqrt{2}\}$ an infinite number of times.) It follows that $\phi(e^{it}/\sqrt{2}) = 0$ for all $t$, hence, the function $\phi$ is identically zero, a contradiction. Thus, $a = 0$ and a similar argument shows that $b = 0$. Thus, $\phi \circ \pi$ is a Blaschke product. Since $\phi$ is not constant, there is a point $\beta$ in $R$ such that $\phi(\beta) = 0$ and therefore $\phi \circ \pi$ must vanish on the infinite set $\pi^{-1}(\beta)$. Thus, $\phi \circ \pi$ is an infinite Blaschke product.

THEOREM 1. *There are two infinite Blaschke products $\phi$ and $\psi$ such that the Banach algebra $\{T_\phi, T_\psi\}''$ is isomorphic to $H^\infty(R)$.*

PROOF. It has been shown by Rudin that there are two inner functions $\phi_1$ and $\psi_1$ which form a separating pair for $A(R)$ [9]. It follows that the map $z \to (\phi_1(z), \psi_1(z))$ is an embedding of the closure of $R$ into the closure of the polydisc $U^2 = \{(z_1, z_2): |z_1| < 1, |z_2| < 1\}$ which takes the boundary of $R$ into the boundary of $U^2$. Let $V$ denote the range of this embedding. Stout has shown that every function $f$ in $A(R)$, when viewed as a function on $V$, can be extended to a continuous function $\tilde{f}$ on the closure of $U^2$ which is analytic on $U^2$ [12, Theorem II.1]. Let $\{p_n\}$ be a sequence of polynomials in two variables which converges uniformly to $\tilde{f}$ on the closure of $U^2$. Then the sequence $\{p_n(\phi_1, \psi_1)\}$ converges uniformly to $f$ on the closure of $R$ which proves that $\phi_1$ and $\psi_1$ generate $A(R)$. Set $\phi = \phi_1 \circ \pi$ and $\psi = \psi_1 \circ \pi$. The theorem now follows from Lemmas 2.2 and 2.3.

To see that Theorem 1 settles Question 1, suppose that $\theta$ is an inner function such that $\{T_\phi\}' \cap \{T_\psi\}' = \{T_\theta\}'$. Then $\{T_\phi, T_\psi\}'' = \{T_\theta\}''$. However, since $T_\theta$ is a unilateral shift, its double commutant $\{T_\theta\}''$ is isomorphic to the Banach algebra $H^\infty(D)$. Thus, according to Theorem 1, the Banach algebras $H^\infty(R)$ and $H^\infty(D)$ are isomorphic. This is a contradiction. In fact, if $R_1$ and $R_2$ are two bounded domains in the plane with analytic boundary and if $H^\infty(R_1)$ is isomorphic to $H^\infty(R_2)$, then $R_1$ and $R_2$ are conformally equivalent by a general result of Chevalley and Kakutani [10].

3. **Reducing subspaces**. We begin with an affirmative result.

THEOREM 2. *If* $\{\chi_\alpha: \alpha \text{ in } \mathcal{Q}\}$ *is a collection of inner functions which contains a finite Blaschke product, if* $\chi$ *is the greatest common divisor of the* $\chi_\alpha$, *and if* $\mathfrak{M}$ *is a closed subspace which reduces each* $T_{\chi_\alpha}$, *then* $\mathfrak{M}$ *reduces* $T_\chi$.

PROOF. The hypotheses imply that there is a finite Blaschke product $\theta$ and inner functions $\psi_\alpha$ such that $\chi_\alpha = \psi_\alpha \circ \theta$ and $\{T_{\chi_\alpha}: \alpha \text{ in } \mathcal{Q}\}' = \{T_\theta\}'$ [13]. Set $\Psi = \text{g.c.d.}\{\Psi_\alpha\}$. It follows that $\chi = \Psi \circ \theta$. In other words

$$(*) \qquad [\text{g.c.d.}\{\Psi_\alpha: \alpha \text{ in } \mathcal{Q}\}] \circ \theta = \text{g.c.d.}\{\Psi_\alpha \circ \theta: \alpha \text{ in } \mathcal{Q}\}.$$

For the case here where the collection $\{\Psi_\alpha\}$ contains a finite Blaschke product, the function $\Psi$ is the Blaschke product vanishing precisely at the common zeroes (counting multiplicities) of the $\Psi_\alpha$. Since $\theta$ is also finite Blaschke, $\Psi \circ \theta$ is the finite Blaschke product which vanishes precisely at the common zeroes (counting multiplicities) of $\Psi_\alpha \circ \theta$, and hence $\Psi \circ \theta = \chi$. The general case of (*) can be shown using Theorem 1 (iv) of [3]. Now suppose that $P$ is a projection which commutes with each $T_{\chi_\alpha}$. Then $P$ commutes with $T_\theta$ and hence with $\Psi(T_\theta)$. But $\Psi(T_\theta) = T_{\Psi \circ \theta} = T_\chi$ which proves the theorem.

Let $A$ be the linear fractional transformation which generates the covering group for $\pi$ as in [1]. Thus, $A$ maps the disk onto itself and a function $\phi$ on the disk is of the form $\Psi \circ \pi$ if and only if $\phi$ is automorphic with respect to $A$, that is, $\phi(A(z)) = \phi(z)$ for all $z$ in $D$. The following lemma deals with a composition operator defined with respect to $A$ which is perturbed in such a way to make the result unitary. It also deals with functions modulus automorphic with respect to $A$. We now define these objects.

The composition operator $C_A$ on $H^2$ is defined by the equation $C_A(f) = f \circ A$. It has been shown by Nordgren [7] that for $f$ in $H^2$,

$$\|C_A(f)\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \operatorname{Re}\left[\frac{e^{i\theta} + A(0)}{e^{i\theta} - A(0)}\right] d\theta.$$

Thus, if $k_A$ is the outer function such that

$$|k_A(e^{i\theta})|^2 = \left[\operatorname{Re}\left[\frac{e^{i\theta} + A(0)}{e^{i\theta} - A(0)}\right]\right]^{-1},$$

then $k_A$ is an invertible element of $H^\infty$ and $\|C_A(k_A f)\|^2 = \|f\|^2$ for all $f$ in $H^2$. Thus, the operator $U_A$ on $H^2$ defined by $U_A(f) = C_A(k_A f)$ is unitary. Let $\phi$ be in $H^\infty$ of the disk, and let $\lambda$ be a scalar of modulus one. The function $\phi$ is said to be modulus automorphic with respect to $A$ of index $\lambda$ if $\phi(A(z)) = \lambda\phi(z)$ for all $z$ in $D$.

LEMMA 3.1. *If* $\phi$ *is modulus automorphic with respect to* $A$ *of index* $\lambda$, *then*

$U_A T_\phi = \lambda T_\phi U_A$. *Hence, the operator* $U_A$ *commutes with* $T_\phi$ *if and only if* $\phi$ *is automorphic with respect to* $A$.

PROOF. Evaluate.

Actually, Lemma 3.1 holds for an arbitrary linear fractional transformation which maps the disk onto itself.

THEOREM 3. *There are two infinite Blaschke products* $\phi$ *and* $\psi$ *and a subspace* $\mathfrak{M}$ *of* $H^2$ *such that* $\mathfrak{M}$ *reduces* $T_\phi$ *and* $T_\psi$ *and* $\mathfrak{M}$ *does not reduce* $T_\chi$ *where* $\chi$ *is the greatest common divisor of* $\phi$ *and* $\psi$.

PROOF. For $a$ in $R$, let $\phi_a$ be the Blaschke product for the set $\pi^{-1}(a)$. It has been shown by Sarason that $\phi_a$ is modulus automorphic with respect to $A$ of index $e^{2\pi t i}$ where $t = \log|a|/\log 2$ [11, p. 18]. Thus, if $a = 1/\sqrt{2}$, then the Blaschke products $\phi = \phi_a \phi_a$ and $\psi = \phi_a \phi_{ia}$ are automorphic with respect to $A$ and their greatest common divisor $\chi = \phi_a$ is modulus automorphic with respect to $A$ of index $-1$. Thus, by Lemma 3.1, the unitary operator $U_A$ commutes with $T_\phi$ and $T_\psi$ and does not commute with $T_\chi$. Since the projections in a $W^*$-algebra always generate the algebra, there is a projection $P$ in the $W^*$-algebra generated by $U_A$ such that $P$ does not commute with $T_\chi$. But this projection does commute with $T_\phi$ and $T_\psi$ which proves the theorem.

The following theorem is closely related to Theorem 3 of [1].

THEOREM 4. *If* $\phi(z) = \pi(z) - \frac{3}{4}$ *and if* $\phi = \chi F$ *is the inner-outer factorization of* $\phi$, *then there is a reducing subspace for* $T_\phi$ *which reduces neither* $T_\chi$ *nor* $T_F$.

PROOF. The function $\chi$ is modulus automorphic and not automorphic with respect to $A$ (see the proof of Theorem 3 in [1]). By Lemma 3.1, the unitary operator $U_A$ does not commute with $T_\chi$ and thus there is a projection $P$ in the $W^*$-algebra generated by $U_A$ such that $P$ does not commute with $T_\chi$. Since $\phi$ is automorphic with respect to $A$, the operator $U_A$ commutes with $T_\phi$ by Lemma 3.1, and thus $P$ commutes with $T_\phi$. From the equations (1) $T_\phi = T_\chi T_F$, (2) $T_\phi P = PT_\phi$, (3) $T_\chi P \neq PT_\chi$, and the fact that (4) $T_F$ is invertible, it follows that $T_F P \neq PT_F$. This completes the proof of Theorem 4.

4. **Comments**. The examples in Theorems 3 and 4 involve projections in the $W^*$-algebra generated by $U_A$ (sometimes called spectral projections for $U_A$). In fact, the operator $U_A$ is a bilateral shift of infinite multiplicity [5] and therefore the $W^*$-algebra generated by $U_A$ is $L^\infty$ of the unit circle.

The spectral subspaces for $U_A$ are reducing subspaces for $T_\pi$, a fact which gives a proof of Theorem 2 in [1] that does not invoke the theory of bundle shifts. The proof of the following proposition involves an analysis of the bundle $E$ of §2 and is omitted. Following Rosenthal [8], an operator $A$ is said to be completely reducible if for each nonzero reducing subspace $\mathfrak{M}$, the operator $A|\mathfrak{M}$ has a nontrivial reducing subspace.

PROPOSITION 4.1. *The operator* $T_\pi$ *is completely reducible.*

This proposition suggests the following reformulation of a question of Nordgren [6] which was shown to be false in general by the first author [1].

*Question.* If $\phi$ is in $H^\infty$ and if $T_\phi$ has a nontrivial reducing subspace $\mathfrak{M}$ such that $T_\phi|\mathfrak{M}$ is irreducible, must there be a function $\psi$ in $H^\infty$ and an inner function $\theta$ which is not a linear fractional transformation such that $\phi = \psi \circ \theta$?

In an abstract of his dissertation, Carl Cowen has announced an affirmative answer to Question 1 if for some $w$ in the unit disk, the greatest common divisor of $\{\chi_\alpha - \chi_\alpha(w)|\alpha \text{ in } \mathcal{C}\}$ is finite Blaschke. It follows that Theorem 2 remains true under this assumption.

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