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# THE REGULAR GROUP $C^*$-ALGEBRA FOR REAL-RANK ONE GROUPS

ROBERT BOYER AND ROBERT MARTIN[1]

ABSTRACT. Let $G$ be a connected semisimple real-rank one Lie group with finite center and let $C^*_\rho(G)$ denote the regular group $C^*$-algebra of $G$. In this paper a complete description of the structure of $C^*_\rho(G)$ is obtained.

1. **Introduction.** Let $G$ be a connected semisimple real-rank one Lie group with finite center and Lie algebra $\mathfrak{g}$. If $G_{\mathbf{C}}$ is the simply connected, complex analytic group corresponding to $\mathfrak{g}_{\mathbf{C}}$, we assume, in addition, that $G$ is the real analytic subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{g}$. Let $C^*_\rho(G)$ denote the regular group $C^*$-algebra of $G$, i.e., the completion of $L_1(G)$ with respect to the norm $\|f\|_\rho = |||\rho(f)|||$ where $\rho$ is the left regular representation of $G$ and $|||\rho(f)|||$ denotes the norm of $\rho(f)$ as a left convolution operator on $L_2(G)$. The purpose of this paper is to give a complete description of the structure of $C^*_\rho(G)$ and thus give a partial answer (one for the above $G$) to a question raised in [6] as to an intrinsic characterization of $C_0(\hat{G})$.

Throughout this paper $H$ will denote a fixed separable infinite-dimensional Hilbert space and $\mathcal{K}(H)$ will denote the compact operators on $H$. We assume, in addition, that $H$ has been identified with $H \oplus H$. When $T$ is a locally compact *Hausdorff* space, we denote by $C^b(T, \mathcal{K}(H))$ the $C^*$-algebra of all norm-continuous bounded functions $t \mapsto x(t)$ of $T$ into $\mathcal{K}(H)$ and by $C^0(T, \mathcal{K}(H))$ the $C^*$-algebra of functions in $C^b(T, \mathcal{K}(H))$ such that $\|x(t)\|$ vanishes at infinity.

The underlying hull-kernel topology on the spectrum of $C^*_\rho(G)$, $\hat{G}_r$, plays a key role in describing the structure of $C^*_\rho(G) \approx C_0(\hat{G})$. The main difficulty occurs when $\hat{G}_r$ is not Hausdorff. When $\hat{G}_r$ is Hausdorff e.g.,

$$G = \operatorname{Spin}(2n+1, 1) \quad \text{for } n \geqslant 1,$$

it follows from [2, 10.9.6] that $C^*_\rho(G)$ is isomorphic to $C^0(\hat{G}_r, \mathcal{K}(H))$. However, when $\hat{G}_r$ is not Hausdorff the above theorem no longer applies and we show, in §3, that it is possible to use the extension theory of C. Delaroche [1] to determine the structure of $C^*_\rho(G)$. We first show that $C^*_\rho(G)$ is isomorphic to the restricted product of certain $C^*$-algebras whose structures

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have concrete descriptions given by [1, Theorem VI.3.8]. Letting $\mathcal{Q}_P$, $\hat{G}_P$, and $\hat{G}_d$ be as in [8, Volume II], it is then a simple matter to give an alternate description of $C^*_\rho(G)$ as the subalgebra of functions in $C^0(\mathcal{Q}_P \cup \hat{G}_d, \mathcal{K}(H))$ which reduce at the points of $\mathcal{Q}_P - \hat{G}_P$ (i.e., the points "responsible" for the non-Hausdorffness of $\hat{G}_r$) by $H \oplus H$.

We refer to [2] and [8] for all undefined terms and notation.

2. **The topology on** $\hat{G}_r$. In this section we summarize the main results concerning the representation theory of $G$ and the topology on $\hat{G}_r$ which we shall need to describe $C^*_\rho(G)$. For a more detailed account we refer to [8, Volume II, Chapter 7 and Epilogue].

Let $G = KAN$ be an Iwasawa decomposition for $G$, $M$ the centralizer of $A$ in $K$, $P = MAN$ (a minimal parabolic subgroup of $G$), and $W = \{1, w\}$ be the Weyl group of $G$, where $w$ is the unique nontrivial element of $W$. We let $\hat{G}_r$ denote the reduced dual of $G$ i.e., the support of $\rho$ in $\hat{G}$.

Up to conjugacy, either $G$ has a unique noncompact Cartan subgroup or $G$ has two Cartan subgroups–one compact and one noncompact. Each conjugacy class of Cartan subgroups makes its own contribution to $\hat{G}_r$. The noncompact Cartan subgroup contributes the collection of irreducible principal series representations, $\hat{G}_P$, together with those irreducibles which arise as summands of reducible principal series representations, $\check{G}_P$. The compact Cartan subgroup contributes the so-called discrete series of $G$, $\hat{G}_d$. Let us briefly recall the parameterizations of these representations given in [8].

If $\mathfrak{a}$ denotes the Lie algebra of $A$, the irreducible unitary representations of $A$ are given by $\lambda^s(\exp H) = \exp(isH)$, $s \in \mathbf{R}$, $H \in \mathfrak{a}$, and so $\hat{A} = \{\lambda^s\colon s \in \mathbf{R}\}$. The hull-kernel topology on $\hat{A}$ agrees with the usual topology it inherits as the character group of the abelian group $A$, i.e., that of $\mathbf{R}$. If $n$ is the dimension of a maximal torus in the compact group $M$, then we may view $\hat{M}$ as a countable discrete subset of $\mathbf{R}^n$ and, hence, $\hat{M} \times \hat{A}$ as a subset of $\mathbf{R}^{n+1}$ with the relative topology. The Weyl group $W$ acts on $\hat{M} \times \hat{A}$ as follows: $1 \cdot (\sigma, s) = (\sigma, s)$ and $w \cdot (\sigma, s) = (w \cdot \sigma, -s)$ where $w \cdot \sigma(m) = \sigma(w^{-1}mw)$, $m \in M$. Under the quotient topology, the orbit space $\mathcal{Q}_P = (\hat{M} \times \hat{A})/W$ is locally compact and Hausdorff.

For $\sigma \in \hat{M}$ and $\lambda^s \in \hat{A}$ we form the finite-dimensional irreducible unitary representation $\sigma \times \lambda^s$ of $P$ via $(\sigma \times \lambda^s)(man) = \sigma(m)\lambda^s(a)$ and write

$$\pi(\sigma, s) = \mathrm{Ind}_P^G \sigma \times \lambda^s.$$

The collection of unitary representations $\{\pi(\sigma, s)\colon \sigma \in \hat{M}, s \in \mathbf{R}\}$ is called the principal series of $G$. It is known that $\pi(\sigma, s)$ is irreducible unless $w \cdot \sigma = \sigma$ and $s = 0$ and in this case $\pi(\sigma, 0)$ may or may not be irreducible (see [5] or [8, Volume I, p. 462]). When $G$ has a unique Cartan subgroup, the results of Wallach [7] show that every member of the principal series is irreducible. If we let $R = \{\sigma \in \hat{M}\colon \pi(\sigma, 0) \text{ is reducible}\}$ and $I = \hat{M} - R$, then for $\sigma \in R$ it is known that $\pi(\sigma, 0)$ decomposes into two inequivalent representations of $G$ (see [4]) which we shall denote by $\pi_\sigma^\pm$. It follows from [8, 5.5.3.3] that $\pi(\sigma, s) \simeq \pi(\sigma', t)$ iff there exists a $c \in W$ such that $c \cdot \sigma = \sigma'$ and $c \cdot s = t$, i.e., $\sigma = \sigma'$ and $s = t$ or $w \cdot \sigma = \sigma'$ and $s = -t$. Thus we may identify the collection of principal series representations with $\mathcal{Q}_P$.

As in [8, Volume II], we let $\hat{G}_P$ denote the subset of $\mathcal{Q}_P$ consisting of irreducible principal series representations. A point $q \in \mathcal{Q}_P - \hat{G}_P$ has coordinates $(\sigma, 0)$ with $\sigma \in R$ and so we may associate to $q$ the *pair* of representations $\pi_\sigma^\pm = \pi_q^\pm$ in $\hat{G}_r$ (it is this association that makes $\hat{G}_r$ non-Hausdorff). Letting $\check{G}_P = \{\pi_q^\pm : q \in \mathcal{Q}_P - \hat{G}_P\}$ we have that $\hat{G}_P \cap \check{G}_P = \varnothing$.

When $G$ has a compact Cartan subgroup (iff rank $G \equiv 1 +$ rank $M =$ rank $K$), one obtains a family of irreducible, square-integrable, unitary representations of $G$ called the discrete series of $G$. Denoting this family by $\hat{G}_d$, we have that $\hat{G}_d \cap (\hat{G}_P \cup \check{G}_P) = \varnothing$ and that it is possible to parameterize $\hat{G}_d$ by a lattice in $\mathbf{R}^{n+1}$ [8, 10.2.4]. Thus we may identify $\hat{G}_d$ with a countable discrete subset of $\mathbf{R}^{n+1}$ which does not intersect $\hat{G}_P \cup \check{G}_P$ and ultimately, $\hat{G}_r$ as a disjoint union of the three subsets $\hat{G}_P$, $\check{G}_P$, and $\hat{G}_d$ of $\mathbf{R}^{n+1}$.

THEOREM (LIPSMAN). *Let $\hat{G}_r$ be the reduced dual of $G$. Then*

(1) *if $G$ has a unique Cartan subgroup, $\hat{G}_r = \hat{G}_P = \mathcal{Q}_P$ and the hull-kernel topology on $\hat{G}_r$ coincides with the natural (Hausdorff) topology of $\mathcal{Q}_P$,*

(2) *if $G$ also has a compact Cartan subgroup, $\hat{G}_r = \hat{G}_P \cup \check{G}_P \cup \hat{G}_d$ (disjoint union) where both $\hat{G}_d$ and $\hat{G}_P \cup \check{G}_P$ are open in $\hat{G}_r$, the topology on $\check{G}_P \cup \hat{G}_d$ is discrete, and the closure of any subset $S \subseteq \hat{G}_P$ consists precisely of those $\pi \in \hat{G}_P \cup \check{G}_P$ which are associated with the points in the natural closure of $S$ in $\mathcal{Q}_P$.*

3. **The structure of** $C_\rho^*(G)$. Since $R = \{\sigma \in \hat{M} : w \cdot \sigma = \sigma\}$ and $I = \hat{M} - R$, we have $\hat{M} \times \hat{A} = (R \times \hat{A}) \cup (I \times \hat{A})$ where both $R \times \hat{A}$, $I \times \hat{A}$ are $W$-invariant and, in fact, $(R \times \hat{A})/W = R \times [0, \infty)$. Let $B = [0, \infty)$ and for $\sigma \in R$ write $B_\sigma = \{(\sigma, s) : s > 0\}$, $\bar{B}_\sigma = \{(\sigma, s) : s \geqslant 0\}$, and $B'_\sigma = B_\sigma \cup \{\pi_\sigma^\pm\}$. Let

$$\mathcal{Q}_R = R \times B = \bigcup_{\sigma \in R} \bar{B}_\sigma, \quad \mathcal{Q}_I = (I \times \hat{A})/W \quad \text{and} \quad B_\theta = \mathcal{Q}_I \cup \hat{G}_d.$$

Then

$$\mathcal{Q}_P = \mathcal{Q}_R \cup \mathcal{Q}_I \quad \text{and} \quad \mathcal{Q}_P \cup \hat{G}_d = \left( \bigcup_{\sigma \in R} \bar{B}_\sigma \right) \cup B_\theta.$$

According to the results of §2, each of the (Hausdorff) fibres $\bar{B}_\sigma$ in $\mathcal{Q}_R$ is associated with the (non-Hausdorff) fibre $B'_\sigma$ in $\hat{G}_r$ where the topology on $B'_\sigma$ is such that as $s \to 0$ in the usual sense, $(\sigma, s)$ approaches both $\pi_\sigma^\pm$ as limit points in $\hat{G}_r$. Thus we may write $\hat{G}_r = (\bigcup_{\sigma \in R} B'_\sigma) \cup B_\theta$.

Let $I_\theta$ denote the ideal in $C_\rho^*(G)$ with $\hat{I}_\theta = B_\theta$ and $I_\sigma$ denote the ideal in $C_\rho^*(G)$ with $\hat{I}_\sigma = B'_\sigma$, $\sigma \in R$. Let $\mathfrak{A}$ be as in [8, Volume II, p. 50]. Then $\mathfrak{A}$ is also a dense selfadjoint subalgebra of $C_\rho^*(G)$ with each element boundedly represented in $\hat{G}_r$. Since each $B_\theta$ and $B'_\sigma$, $\sigma \in R$, is both open and closed in $\hat{G}_r$, each $I_\sigma$, $\sigma \in R \cup \{\theta\}$, is a direct summand of $C_\rho^*(G)$. So for $\sigma \in R \cup$

$\{\theta\}$ we may let $\mathfrak{A}_\sigma$ denote the canonical image of $\mathfrak{A}$ in $I_\sigma$. Then $\mathfrak{A}_\sigma$ is a dense selfadjoint subalgebra of $I_\sigma$ having the property that each of its elements is boundedly represented in $\hat{I}_\sigma$. We now use the extension theory of Delaroche to give concrete descriptions of these ideals and then prove that $C_\rho^*(G)$ is isomorphic to the restricted product [2, 1.9.14] of these ideals.

PROPOSITION 1. (i) *Let $\sigma \in R$. Then $I_\sigma$ is isomorphic to the $C^*$-algebra of pairs*

$$(m, (c_1, c_2)) \in C^b(B_\sigma, \mathcal{K}(H)) \times (\mathcal{K}(H) \oplus \mathcal{K}(H))$$

*such that* $\lim_{t\to\infty} m(\sigma, t) = 0$ *and* $\lim_{t\to 0} m(\sigma, t) = (c_1, c_2)$.

(ii) *$I_\theta$ is isomorphic to $C^0(B_\theta, \mathcal{K}(H))$.*

PROOFS. (i) For $\sigma \in R$, let $J_\sigma$ be the ideal of $I_\sigma$ with $\hat{J}_\sigma = B_\sigma$. From [8, Volume II, p. 50] it follows that $J_\sigma$ is a $C^*$-algebra with continuous trace [2, 4.5.2]. Since $H^3(B_\sigma, \mathbf{Z}) = 0$, it follows from [2, 10.9.6] that $J_\sigma$ is isomorphic to $C^0(B_\sigma, \mathcal{K}(H))$. Now $I_\sigma$ is isomorphic to an extension of $C^0(B_\sigma, \mathcal{K}(H))$ by $\mathcal{K}(H) \oplus \mathcal{K}(H)$, in fact, using [1, Theorem VI.3.8], one can concretely describe $I_\sigma$ as above once the positive integers $m$ and $n$ are determined in the equation

$$\lim_{t\to 0} \operatorname{tr} \pi(\sigma, t)(f) = m \operatorname{tr} \pi_\sigma^+(f) + n \operatorname{tr} \pi_\sigma^-(f), \qquad f \in \mathfrak{A}_\sigma.$$

However, the results of [8, Volume II, p. 50] show that $m = n = 1$ and so (i) follows.

(ii) Since $\hat{I}_\theta$ is Hausdorff and $H^3(I_\theta, \mathbf{Z}) = 0$, (ii) follows from [2, 10.9.6] since [8, Volume II, pp. 50, 422] shows that $I_\theta$ is a $C^*$-algebra with continuous trace.

PROPOSITION 2. *Let $\sigma \in R$. Then $I_\sigma$ is isomorphic to the subalgebra $D$ of functions in $C^0(\overline{B}_\sigma, \mathcal{K}(H))$ which reduce at $(\sigma, 0)$ by $H \oplus H$.*

PROOF. For $f \in D$, the pair $(m, (c_1, c_2))$ where $m(\sigma, t) = f(\sigma, t)$ for $t \in (0, \infty)$ and $(c_1, c_2) = f(\sigma, 0)$ is clearly in $I_\sigma$. Since the mapping $f \mapsto (m, (c_1, c_2))$ is an isomorphicm of $D$ onto $I_\sigma$, the proposition follows.

LEMMA 1. *Let $\mathfrak{a}$ be a $C^*$-algebra without identity. If $\hat{\mathfrak{a}} = \bigcup_1^\infty X_n$ where the $X_n$ are disjoint nonempty open subsets of $\hat{\mathfrak{a}}$, then $\mathfrak{a}$ is isomorphic to the restricted product $B$ of the ideals $I_n$, where $\hat{I}_n = X_n$.*

PROOF. Let $C = \bigcup_{k=1}^\infty \bigoplus_{n=1}^k I_n$ and consider the ideal $J = \overline{C}$ of $\mathfrak{a}$. It is easy to see that for any $\pi \in \hat{\mathfrak{a}}$, $\pi(J) \neq 0$. Thus $J = \mathfrak{a}$ by [2, 3.2.2]. We now map $C$ onto a dense subset of $B$ in the obvious way. Since this mapping is an isometry, it extends to an isomorphism of $\mathfrak{a}$ onto $B$.

THEOREM 1. *$C_\rho^*(G)$ is isomorphic to the restricted product of the $C^*$-algebras $I_\sigma$, $\sigma \in R \cup \{\theta\}$.*

PROOF. Since $\widehat{C_\rho^*(G)} = \hat{G}_r = (\bigcup_{\sigma\in R} B_\sigma') \cup B_\theta$, this is immediate from Lemma 1.

THEOREM 2. *$C_\rho^*(G)$ is isomorphic to the subalgebra of $C^0(\mathcal{Q}_P \cup \hat{G}_d, \mathcal{K}(H))$ of functions which reduce at the points of $\mathcal{Q}_P - \hat{G}_P$ by $H \oplus H$. In particular, when $\check{G}_P = \varnothing$, $C_\rho^*(G)$ is isomorphic to $C^0(\hat{G}_r, \mathcal{K}(H))$.*

PROOF. By Theorem 1 we have that $C_\rho^*(G)$ is isomorphic to the restricted product $P$ of the ideals $I_\sigma$, $\sigma \in R \cup \{\theta\}$ whose structures are given by Propositions 1(ii) and 2. For $f = \{f_\sigma\}$ in $P$ we define the function $F$ on $\mathcal{Q}_P \cup \hat{G}_d = (\bigcup_{\sigma \in R} \bar{B}_\sigma) \cup B_\theta$ by $F(v) = f_\sigma(v)$ if $v \in \bar{B}_\sigma$, $\sigma \in R$, and $F(v) = f_\theta(v)$ if $v \in B_\theta$. Then $F$ is easily seen to be a norm-continuous bounded function on the Hausdorff space $\mathcal{Q}_P \cup \hat{G}_d$ for which $\|F(t)\|$ vanishes at infinity and $F(\sigma, 0) = f_\sigma(0, 0) = (c_1(\sigma), c_2(\sigma))$ for $\sigma \in R$. Theorem 2 now follows since the mapping $f \mapsto F$ is an isomorphism of $P$ onto the above subalgebra.

4. **Some examples**. A. If $G = \mathrm{Spin}(2n + 1, 1)$ for $n \geqslant 1$, then $\check{G}_P = \varnothing$ (see [5] or [7]). Since $\hat{G}_d = \varnothing$, we have $\hat{G}_r = \hat{G}_P$ is Hausdorff and

$$C_\rho^*(G) \approx C^0\big(\hat{G}_r, \mathcal{K}(H)\big).$$

B. For $G = \mathrm{SL}(2, \mathbf{R})$, $M = \{\pm e\}$ and we may take $\hat{M} = \{0, 1\}$ with $R = \{1\}$ and $I = \{0\}$. Thus we may identify $\hat{G}_r$ with the following subset of $\mathbf{R}^2$: $\hat{G}_P$ consists of the two fibres $\{(0, s): s \geqslant 0\}$ and $\{(1, s): s > 0\}$; $\check{G}_P$ is a pair of points at $(1, -\frac{1}{2})$; and $\hat{G}_d$ consists of the infinite collection of pairs of points at $(-1, -n)$, $n = 1, \frac{3}{2}, 2, \ldots$. The hull-kernel topology on $\hat{G}_r$ is then the relative topology $\hat{G}_r$ obtains as a subset of $\mathbf{R}^2$ with the one exception that as $(1, s) \to (1, 0)$ in the usual sense, $(1, s)$ approaches the pair of points at $(1, -\frac{1}{2})$ as limit points. To describe $C_\rho^*(G)$ we let $X = \{(0, s): s \geqslant 0\} \cup \{(1, s): s \geqslant 0\} \cup \hat{G}_d$ with the relative topology of $\mathbf{R}^2$. $C_\rho^*(G)$ is then isomorphic to the subalgebra of $C^0(X, \mathcal{K}(H))$ consisting of functions which reduce at $(1, 0)$ by $H \oplus H$.

C. For $G = \mathrm{Spin}(4, 1)$, $M = \mathrm{Spin}(3) \approx \mathrm{SU}(2)$ and we may parameterize $\hat{M}$ by nonnegative half-integers with $R = \{\frac{1}{2}, \frac{3}{2}, \ldots\}$, $I = \{0, 1, 2, \ldots\}$. Using the results of Dixmier [3], we may parameterize $\hat{G}_d$ by pairs of points at $(n, -q)$ where $n = 1, \frac{3}{2}, 2, \ldots$ and $q = n, n - 1, \ldots, \frac{3}{2}$ or 1. Thus $\hat{G}_r$ can be identified with the following subset of $\mathbf{R}^2$: $\hat{G}_P$ is the collection of fibres $\{(n, s): s \geqslant 0$ if $n \in I$ and $s > 0$ if $n \in R\}$; $\check{G}_P$ is the infinite collection of pairs of points at $(n, -\frac{1}{2})$, $n \in R$; and $\hat{G}_d$ consists of the infinite collection of pairs of points at $(n, -q)$, $n = 1, \frac{3}{2}, \ldots$ and $q = n, n - 1, \ldots, \frac{3}{2}$ or 1. Since $\check{G}_P \neq \varnothing$, $\hat{G}_r$ is not Hausdorff. To describe $C_\rho^*(G)$ we let $X = \bigcup_{n \in R \cup I}\{(n, s): s \geqslant 0\} \cup \hat{G}_d$ with the relative topology of $\mathbf{R}^2$. Then $C_\rho^*(G)$ is isomorphic to the subalgebra of $C^0(X, \mathcal{K}(H))$ consisting of functions which reduce at the points $(n, 0)$, $n \in R$, by $H \oplus H$.

D. Let $G = \mathrm{SO}_e(n, 1)$, $n \geqslant 2$, and $G'$ be the two-fold covering of $G$–so $G' = \mathrm{SL}(2, \mathbf{R})$ for $n = 2$ and $G' = \mathrm{Spin}(n, 1)$ for $n \geqslant 3$. $G'$ then satisfies the hypotheses of this paper. From [5] we know that even though $G'$ may have reducible principal series (iff $n$ is even), $G$ does not. Since $\hat{G}_r \subseteq \hat{G}'_r$ has the relative hull-kernel topology, we see that $\hat{G}_r$ is Hausdorff [for example, if

$n = 2$ and $D$ denotes the subset of $\hat{G}_d$ (in $B$) consisting of pairs of points at $(-1, -n)$, $n = 1, 2, \ldots$, then $\hat{G}_r = \{(0, s): s \geqslant 0\} \cup D$, while if $n = 4$ and $D$ denotes the subset of $\hat{G}_d$ (in $C$) of pairs of points at $(n, -q)$, $n \in I$, then $\hat{G}_r = \bigcup_{n \in I}\{(n, s): s \geqslant 0\} \cup D]$. Thus it follows, as in the proof of Proposition 1(ii), that $C_\rho^*(G) \approx C^0(\hat{G}_r, \mathcal{K}(H))$.

5. **A remark on** $C_0(\hat{G})$. When $G$ is a locally compact abelian group, it is common to denote the collection of continuous functions on the dual group $\hat{G}$ which vanish at infinity by $C_0(\hat{G})$. In a recent paper [6], R. Lipsman defined an analogue of this space for separable locally compact unimodular type I groups as follows: letting $dg$ denote Haar measure on $G$,

$$\mathcal{F}f(\pi) = \hat{f}(\pi) = \int f(g)\pi(g)\,dg$$

be the Fourier-transform of $f \in L_1(G)$, $|||\hat{f}(\pi)|||$ the operator norm of $\hat{f}(\pi)$, $\|\hat{f}\|_\infty = \operatorname{ess\,sup}_{\pi \in \hat{G}}|||\hat{f}(\pi)|||$ (with respect to Plancherel measure on $\hat{G}$), and $A(\hat{G}) = \mathcal{F}(L_1(G))$, then $C_0(\hat{G})$ is defined to be the closure of the algebra $A(\hat{G})$ with respect to the norm $\|\cdot\|_\infty$. The question is then raised as to determining an intrinsic characterization of $C_0(\hat{G})$. Since $C_0(\hat{G})$ is easily seen to be isomorphic to $C_\rho^*(G)$ (see [6]), the results of this paper seem to indicate that this will be a difficult problem and that the hull-kernel topology on the spectrum of $C_\rho^*(G)$, $\hat{G}_r$, will play a key role in determining an intrinsic characterization of $C_0(\hat{G})$. In fact, Theorem 2 shows that when $G$ is as in the introduction of this paper, an intrinsic characterization of $C_0(\hat{G})$ must take the non-Hausdorff nature of $\hat{G}_r$ into consideration.

We also remark that for amenable groups, $C_0(\hat{G}) \approx C_\rho^*(G) \approx C^*(G)$ (the group $C^*$-algebra of $G$), and although it is quite easy to describe $C_0(\hat{G})$ for abelian or compact groups, we know of no other separable unimodular type I amenable group for which the structure of $C^*(G)$ has been determined.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19174 (Current address of Robert Boyer)

*Current address* (Robert Martin): Department of Mathematics, Middlebury College, Middlebury, Vermont 05753