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A SHORT PROOF OF THE FOURIER INVERSION FORMULA

ALAIN ROBERT

ABSTRACT. We give an elementary proof of the Fourier inversion formula (on the real line) based on the Poisson summation formula.

Let f be a complex valued Schwartz function (i.e. all derivatives of f decrease rapidly) on the real line. Its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\, dx$$

(again in the Schwartz class), and we intend to prove the inversion $\hat{\hat{f}}(x) = f(-x)$. Because $(\tau_a f)\hat{}\,(y) = e^{2\pi i a y}\hat{f}(y)$, $f(x+a)\hat{\hat{}} = \hat{\hat{f}}(x-a)$ and it is enough to prove the inversion formula at the origin: $\hat{\hat{f}}(0) = f(0)$. (Here, $(\tau_a f)(x) = f(x+a)$.)

1. The series $\sum_{m=-\infty}^{\infty} f(x+m)$ converges uniformly. Its sum is a continuous periodic function F having Fourier coefficients

$$c_n(F) = \int_0^1 F(x) e^{-2\pi i n x}\, dx = \sum_n \int_0^1 f(x+m) e^{-2\pi i n(x+m)}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x}\, dx = \hat{f}(n) \quad \text{(quickly decreasing)}.$$

Hence the Fourier series $\sum \hat{f}(n) e^{2\pi i n x}$ converges uniformly to F and $x = 0$ gives the *Poisson summation formula* $\sum_n f(n) = \sum_m \hat{f}(m)$.

2. We apply the Poisson formula to $f_k(x) = f(kx)$ $(k \neq 0)$ and get

$$\sum_n f(kn) = \sum_m \hat{f}(m/k) \cdot 1/|k|.$$

Letting $k \to \infty$, the left-hand side tends to $f(0)$ whereas the right-hand side is a Riemann sum approaching the integral $\int_{-\infty}^{\infty} \hat{f}(y)\, dy = \hat{\hat{f}}(0)$.

3. One can also use the Poisson formula

$$\sum f(n) = \sum \hat{f}(n) = \sum \hat{\hat{f}}(n)$$

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with the new function $e^{2\pi iax}f$ getting

$$\sum f(n)e^{2\pi ian} = \sum \hat{\hat{f}}(n)e^{-2\pi ian} = \sum \hat{\hat{f}}(-n)e^{2\pi ian}$$

(for all real values of a). The identity of these continuous functions of a implies the equality of their Fourier coefficients $f(n) = \hat{\hat{f}}(-n)$. This last method is essentially due to Gel'fand (cf. [1]).

The reader will observe that no hard theorem in integration is needed in these derivations, and the same pattern works as well over the p-adic fields $\mathbf{Q}_p$ instead of $\mathbf{R}$.

References

1. N. J. Vilenkin, *Special functions and the theory of group representations*, "Nauka", Moscow, 1965; English transl., Transl. Math. Monographs, vol. 22, Chap. II, p. 3, Amer. Math. Soc., Providence, R.I., 1968. MR **35** #420; **37** #5429.

Institut de Mathématiques, Chantemerle 20, CH-2000, Neuchatel, Switzerland