A SHORT PROOF OF THE FOURIER INVERSION FORMULA

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Abstract. We give an elementary proof of the Fourier inversion formula (on the real line) based on the Poisson summation formula.

Let \( f \) be a complex valued Schwartz function (i.e. all derivatives of \( f \) decrease rapidly) on the real line. Its Fourier transform \( \hat{f} \) is defined by

\[
\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i xy} \, dx
\]

(again in the Schwartz class), and we intend to prove the inversion \( \hat{f}(x) = f(-x) \). Because \( (\tau_a f)(y) = e^{2\pi iay} \hat{f}(y), f(x + a) = \hat{f}(x - a) \) and it is enough to prove the inversion formula at the origin: \( \hat{f}(0) = f(0) \). (Here, \( (\tau_a f)(x) = f(x + a) \).)

1. The series \( \sum_{m=-\infty}^{\infty} f(x + m) \) converges uniformly. Its sum is a continuous periodic function \( F \) having Fourier coefficients

\[
c_n(F) = \int_{0}^{1} F(x) e^{-2\pi inx} \, dx = \sum_{m} \int_{0}^{1} f(x + m) e^{-2\pi in(x+m)} \, dx
\]

\[
= \int_{-\infty}^{\infty} f(x) e^{-2\pi inx} \, dx = \hat{f}(n) \quad \text{(quickly decreasing)}.
\]

Hence the Fourier series \( \sum \hat{f}(n) e^{2\pi inf} \) converges uniformly to \( F \) and \( x = 0 \) gives the Poisson summation formula \( \sum_{n} f(n) = \sum_{m} \hat{f}(m) \).

2. We apply the Poisson formula to \( f_k(x) = f(kx) \) \((k \neq 0)\) and get

\[
\sum_{n} f(kn) = \sum_{m} \hat{f}(m/k) \cdot 1/|k|.
\]

Letting \( k \to \infty \), the left-hand side tends to \( f(0) \) whereas the right-hand side is a Riemann sum approaching the integral \( \int_{-\infty}^{\infty} \hat{f}(y) \, dy = \hat{f}(0) \).

3. One can also use the Poisson formula

\[
\sum f(n) = \sum \hat{f}(n) = \sum \hat{f}(n)
\]
with the new function $e^{2\pi i a x}$ getting

$$\sum f(n)e^{2\pi i an} = \sum \hat{f}(n)e^{-2\pi i an} = \sum \hat{f}(-n)e^{2\pi i an}$$

(for all real values of $a$). The identity of these continuous functions of $a$ implies the equality of their Fourier coefficients $f(n) = \hat{f}(-n)$. This last method is essentially due to Gel'fand (cf. [1]).

The reader will observe that no hard theorem in integration is needed in these derivations, and the same pattern works as well over the $p$-adic fields $\mathbb{Q}_p$ instead of $\mathbb{R}$.

REFERENCES