

A SHORT PROOF OF THE FOURIER INVERSION FORMULA

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ABSTRACT. We give an elementary proof of the Fourier inversion formula (on the real line) based on the Poisson summation formula.

Let f be a complex valued Schwartz function (i.e. all derivatives of f decrease rapidly) on the real line. Its Fourier transform \hat{f} is defined by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy} dx$$

(again in the Schwartz class), and we intend to prove the inversion $\hat{\hat{f}}(x) = f(-x)$. Because $(\tau_a f)^\wedge(y) = e^{2\pi iay} \hat{f}(y)$, $f(x+a)^\wedge = \hat{f}(x-a)$ and it is enough to prove the inversion formula at the origin: $\hat{\hat{f}}(0) = f(0)$. (Here, $(\tau_a f)(x) = f(x+a)$.)

1. The series $\sum_{m=-\infty}^{\infty} f(x+m)$ converges uniformly. Its sum is a continuous periodic function F having Fourier coefficients

$$\begin{aligned} c_n(F) &= \int_0^1 F(x)e^{-2\pi inx} dx = \sum_n \int_0^1 f(x+m)e^{-2\pi in(x+m)} dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi inx} dx = \hat{f}(n) \quad (\text{quickly decreasing}). \end{aligned}$$

Hence the Fourier series $\sum \hat{f}(n)e^{2\pi inx}$ converges uniformly to F and $x=0$ gives the *Poisson summation formula* $\sum_n f(n) = \sum_m \hat{f}(m)$.

2. We apply the Poisson formula to $f_k(x) = f(kx)$ ($k \neq 0$) and get

$$\sum_n f(kn) = \sum_m \hat{f}(m/k) \cdot 1/|k|.$$

Letting $k \rightarrow \infty$, the left-hand side tends to $f(0)$ whereas the right-hand side is a Riemann sum approaching the integral $\int_{-\infty}^{\infty} \hat{f}(y) dy = \hat{\hat{f}}(0)$.

3. One can also use the Poisson formula

$$\sum f(n) = \sum \hat{f}(n) = \sum \hat{\hat{f}}(n)$$

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with the new function $e^{2\pi iax} f$ getting

$$\sum f(n)e^{2\pi ian} = \sum \hat{f}(n)e^{-2\pi ian} = \sum \hat{f}(-n)e^{2\pi ian}$$

(for all real values of a). The identity of these continuous functions of a implies the equality of their Fourier coefficients $f(n) = \hat{f}(-n)$. This last method is essentially due to Gel'fand (cf. [1]).

The reader will observe that no hard theorem in integration is needed in these derivations, and the same pattern works as well over the p -adic fields \mathbf{Q}_p instead of \mathbf{R} .

REFERENCES

1. N. J. Vilenkin, *Special functions and the theory of group representations*, "Nauka", Moscow, 1965; English transl., Transl. Math. Monographs, vol. 22, Chap. II, p. 3, Amer. Math. Soc., Providence, R.I., 1968. MR 35 #420; 37 #5429.

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