A NOTE ON RIESZ OPERATORS

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Abstract. The purpose of this note is to settle a problem posed by Caradus, Pfaffenberger, and Yood; namely, it is proved that every Riesz operator $R$ on a Hilbert space has a decomposition $R = C + Q$ where $C$ is compact and both $Q$ and $CQ - QC$ are quasinilpotent.

Let $H$ denote a complex, separable, infinite dimensional Hilbert space. In [5], West showed that every Riesz operator was a decomposable Riesz operator, i.e., $R = C + Q$ where $C$ is a compact operator and $Q$ is quasinilpotent. In general, this decomposition is not unique.

A Riesz operator is said to be fully decomposable if $R$ is decomposable and, in addition, $C$ commutes with $Q$ for some decomposition $C$ and $Q$.

In [1, p. 58], an example of Gillespie and West was given showing that there are some Riesz operators on $H$ which are not fully decomposable. They produced a Riesz operator $R$ for which no decomposition could commute. This leads to the following question proposed in [1, p. 59]: Can every Riesz operator be decomposed in such a manner that the commutator $CQ - QC$ is quasinilpotent? The purpose of this note is to give a positive answer to this question, and, in fact, a slightly stronger result is proved.

The key to our proof is a lemma of Gohberg and Krein which was stated without proof in [3, p. 17] and was later stated and proved by Stampfli [4].

Lemma 1 [Gohberg-Krein, Stampfli]. Let $E$ be a closed set in $C$. Let $\sigma(T) \setminus E$ consist of isolated points $\{\lambda_j\}$ which of necessity cluster only on $E$. Let each $\lambda_j$ be a point of finite multiplicity. Then, $T = S + K$ where $K$ is compact and $\sigma(S) \subseteq E$.

Our theorem will depend heavily on the "Stampfli decomposition" and on its notation. Let us recall the pertinent steps. It was shown by Stampfli [4] that for a $T$ satisfying the hypotheses of Lemma 1,
That is, if
\[ P_j = \frac{1}{2\pi i} \int_{|\lambda - \lambda_j| = \epsilon_j} (\lambda - T)^{-1} d\lambda \]
where \( 0 < \epsilon_j < \min\{\min_{i \neq j}(|\lambda_i - \lambda_j|), \text{dist}(\lambda_j, E)\} \) for \( i \neq j \), then \( T \) has the matrix form listed above, where \( L = QTQ \) and \( Q \) is the orthogonal projection on \((\Sigma P_j H)^{\perp}\). Now let \( \{\alpha_k\} \) be a countable dense subset of \( E \). With each \( \lambda_j \), associate an \( \alpha_k \) as follows. Choose \( \alpha_k \) such that \( |\alpha_k - \lambda_j| < 2 \text{dist}(\lambda_j, E) \). For simplicity write \( \alpha_k \) as \( \alpha_j \). Next set
\[
K = \begin{pmatrix}
\lambda_1 - \alpha_1 & \cdots & 0 \\
0 & \lambda_2 - \alpha_2 & \cdots \\
& 0 & \ddots & \vdots \\
0 & \cdots & 0 & \lambda_n - \alpha_n
\end{pmatrix}
\]
and define \( S = T - K \). Obviously, \( K \) is compact.

We are now ready to state our theorem.

**Theorem 1.** Let \( T \) satisfy the hypotheses of Lemma 1. Then in the “Stampfli decomposition” \( T = S + K \), the commutator \( SK - KS \) is a compact quasi-nilpotent operator.

**Proof.** Clearly \( SK - KS \) is compact. Using the above notation, it is easily seen that
\[
SK - KS = \begin{pmatrix}
\hat{S}\hat{K} - \hat{K}\hat{S} & \hat{K}^* \\
0 & 0
\end{pmatrix}
\]
where \( \hat{K}^* \) denotes the product of \( \hat{K} \) and the northeast block of \( S \) and \( \hat{S} \) denotes the northwest corner of \( S \).

We first show that \( \hat{S}\hat{K} - \hat{K}\hat{S} \), viewed as an operator on \( \Sigma P_j H \) is quasi-nilpotent. As a matrix \( \hat{S}\hat{K} - \hat{K}\hat{S} \) is a compact operator, upper triangular with
main diagonal identically zero. Let \( K_n = P_n(\hat{S} \hat{K} - \hat{K} \hat{S})P_n \) where \( P_n \) projects onto \( \text{sp}\{e_1, \ldots, e_n\} \) where \( \{e_j\}_{j=1}^{\infty} \) is the orthonormal basis for which \( \hat{S} \) is upper triangular. Since \( P_n \) converges to the identity in the strong operator topology, \( K_n \) converges uniformly to \( \hat{S} \hat{K} - \hat{K} \hat{S} \). Clearly, each \( K_n \) is quasinilpotent (actually nilpotent), so by [3, Theorem 4.1], \( \hat{S} \hat{K} - \hat{K} \hat{S} \), as a uniform limit of compact quasinilpotent operators, is quasinilpotent.

To complete the proof, it suffices to show that \( SK - KS \) is quasinilpotent. By the Riesz spectral theorem for compact operators, this is equivalent to showing that \( SK - KS \) has no nonzero eigenvalues.

So assume \( \lambda \neq 0 \) and

\[
\begin{pmatrix}
\hat{S} \hat{K} - \hat{K} \hat{S} & \hat{K}^* \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2
\end{pmatrix} = \lambda
\begin{pmatrix}
e_1 \\
e_2
\end{pmatrix}.
\]

Upon equating components of the vectors, we see that

\[e_2 = 0 \quad \text{and} \quad (\hat{S} \hat{K} - \hat{K} \hat{S})e_1 = \lambda e_1\]

which is impossible; thus \( SK - KS \) is quasinilpotent. This completes the proof.

**Corollary.** For a Riesz operator \( R \) on a Hilbert space, we have \( R = C + Q \) where \( C \) is compact and both \( Q \) and \( CQ - QC \) are quasinilpotent.

**References**


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