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SPECTRAL PROPERTIES OF LINEAR OPERATORS FOR WHICH T^*T AND $T + T^*$ COMMUTE

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ABSTRACT. The class of linear operators for which T^*T and $T + T^*$ commute is studied. It is shown that such operators are normaloid. If T is also completely nonnormal, then $\sigma(T) = \sigma(T^*)$. Also, isolated points of $\sigma(T)$ are reducing eigenvalues. Finally, if $\sigma(T)$ is a subset of either a vertical line or the real axis, then T is normal.

1. **Introduction.** Bounded linear operators T such that T^*T and $T + T^*$ commute have been studied in [4], [5], and [6]. The set of such operators is denoted by θ [4]. Embry has shown that if $T \in \theta$ and T is not normal, then $\sigma(T) \cap \sigma(T^*) \neq \varnothing$ [6]. We shall show that if T is completely nonnormal and $T \in \theta$, then $\sigma(T) = \sigma(T^*)$. We shall also show that isolated points of $\sigma(T)$ are eigenvalues and operators in θ are normaloid.

While parts of this paper provide generalizations of some of the results of [4], the results of this paper tend to be of a different nature than those of [4]. The techniques used here are also different.

2. **Notation and preliminary results.** The notation used here is consistent with [4]. All operators are bounded, linear, and act on a separable Hilbert space $\mathcal{H}$. For operators X, Y we let $[X, Y] = XY - YX$. Then $\theta = \{T\colon [T^*T, T + T^*] = 0\}$. Let

$$B(\lambda) = (\lambda - T^*)(\lambda - T) = \lambda^2 - \lambda(T^* + T) + T^*T.$$

For $T \in \theta$, and any value of λ, $B(\lambda)$ is normal. Let E be the spectral measure associated with the algebra generated by T^*T and $T + T^*$. Then

$$T^*T = \int_\Delta g(s)E(ds), \qquad T + T^* = \int_\Delta h(s)E(ds),$$

Δ a compact subset of the plane. The set of λ for which $B(\lambda)$ is not invertible is denoted $\hat{\sigma}(B)$. Clearly $\lambda \in \hat{\sigma}(B)$ if and only if $\bar{\lambda} \in \hat{\sigma}(B)$. For a set S, ∂S denotes its boundary.

PROPOSITION 1. *If $T \in \theta$, then* $\partial\sigma(T^*) \cup \partial\sigma(T) \subseteq \hat{\sigma}(B) \subseteq \sigma(T) \cup \sigma(T^*)$.

PROOF. The second inclusion is obvious. If $\lambda \in \partial\sigma(T)$, then λ is in the

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approximate point spectrum of T. Thus there exist $\phi_n \in \mathcal{H}$ such that $B(\lambda)\phi_n \to 0$, $\|\phi_n\| = 1$. Hence $\lambda \in \hat{\sigma}(B)$. If $\lambda \in \partial\sigma(T^*)$, then $\bar{\lambda} \in \partial\sigma(T)$. Thus $\bar{\lambda} \in \hat{\sigma}(B)$ and $\lambda \in \hat{\sigma}(B)$ as desired. □

We note that both inclusions in Proposition 1 may be proper for completely nonnormal $T \in \theta$. For example, if T is the unilateral shift, $\hat{\sigma}(B)$ is the unit circle while $\sigma(T)$ is the unit disc. In this case, $\partial\sigma(T) = \hat{\sigma}(B)$.

Before being able to finish the development of our basic definitions, we need a fundamental fact about operators in θ.

PROPOSITION 2. *If $T \in \theta$, then $4T^*T - (T^* + T)^2 \geqslant 0$.*

PROOF. Suppose that $4T^*T - (T^* + T)^2 \geqslant 0$ is not true. Let $\Delta = \{s: h^2(s) - 4g(s) > 0\}$. Then $E(\Delta) > 0$. Take $\lambda_0 \in \Delta$ such that $h(\lambda_0)$, $g(\lambda_0)$ are in the essential ranges of h and g respectively.

Let

$$\lambda_1 = \frac{h(\lambda_0) + \sqrt{h^2(\lambda_0) - 4g(\lambda_0)}}{2} \quad \text{and} \quad \lambda_2 = \frac{h(\lambda_0) - \sqrt{h^2(\lambda_0) - 4g(\lambda_0)}}{2}.$$

Note that $\lambda^2 - \lambda h(\lambda_0) + g(\lambda_0)$ has λ_1, λ_2 as two distinct real roots. Let $\Delta_1 \subseteq \Delta$ be such that $E(\Delta_1) > 0$ and $h(\lambda)$ is close to $h(\lambda_0)$, $g(\lambda)$ close to $g(\lambda_0)$ for all $\lambda \in \Delta_1$. Then $\lambda_i^2 - \lambda_i h(\lambda) + g(\lambda)$ is small for all $\lambda \in \Delta_1$. Thus

$$B(\lambda_i)E(\Delta_1) = \int_{\Delta_1} (\lambda_i^2 - \lambda_i h(s) + g(s))E(ds)$$

is small in norm for $i = 1, 2$.

Hence if $\phi \in R(E(\Delta_1))$, the range of $E(\Delta_1)$, and $\|\phi\| = 1$, we have

$$\|(\lambda_i - T)\phi\|^2 = ((\lambda_i - T^*)(\lambda_i - T)\phi, \phi) = (B(\lambda_i)\phi, \phi)$$

is small for $i = 1, 2$. But $\lambda_1 \neq \lambda_2$ so this is a contradiction and $E(\Delta) = 0$ as desired. □

For $T \in \theta$, let

$$C = ((T^* + T) + i\sqrt{4T^*T - (T^* + T)^2})/2. \tag{1}$$

From (1) and Proposition 2 we have $C + C^* = T + T^*$, $C^*C = T^*T$, $B(\lambda) = (\lambda - C^*)(\lambda - C)$, C is normal, and $\hat{\sigma}(B) = \sigma(C) \cup \sigma(C^*)$.

$\sigma(C)$ is contained in the closed upper half plane. The spectral measure associated with C will be denoted by F so that $C = \int_{\sigma(C)} sF(ds)$.

3. **Operators in θ are normaloid.** We will now develop several useful facts about operators in θ. The real numbers are denoted by $\mathfrak{R}$.

THEOREM 1. *If $T \in \theta$, then $F(\mathfrak{R})$ reduces T and $TF(\mathfrak{R})$ is hermitian.*

PROOF. Partition $[-\|T\|, \|T\|]$ into n equal pieces of length $2\|T\|/n$. Let λ_i be the midpoint of the ith piece, F_i the associated spectral projection of the ith piece. Then $\|(C - \lambda_i)F_i\| \leqslant \|T\|/n$. But

$$\begin{aligned}\|T\|^2/n^2 &\geqslant \|(C-\lambda_i)F_i\phi\|^2 = ((C-\lambda_i)F_i\phi,(C-\lambda_i)F_i\phi)\\ &= ((C^*-\lambda_i)(C-\lambda_i)F_i\phi, F_i\phi) = ((T^*-\lambda_i)(T-\lambda_i)F_i\phi, F_i\phi)\\ &= \|(T-\lambda_i)F_i\phi\|^2.\end{aligned}$$

Thus $\|(T-\lambda_i)F_i\| \leqslant \|T\|/n$. Hence, $\|(C-T)F_i\| \leqslant 2\|T\|/n$. But then for any $\phi \in \mathcal{H}$,

$$\begin{aligned}\|(C-T)F(\mathfrak{R})\phi\| &\leqslant \sum_{i=1}^{n} \|(C-T)F_i\phi\| \leqslant \sum_{i=1}^{n} \frac{2\|T\|}{n}\|F_i\phi\|\\ &\leqslant \left(\sum_{i=1}^{n}\left(\frac{2\|T\|}{n}\right)^2\right)^{1/2}\left(\sum_{i=1}^{n}\|F_i\phi\|^2\right)^{1/2} = \frac{2\|T\|}{\sqrt{n}}\|F(\mathfrak{R})\phi\|.\end{aligned}$$

Thus $TF(\mathfrak{R}) = CF(\mathfrak{R})$. But $T + T^* = C + C^*$ so that $C^*F(\mathfrak{R}) = T^*F(\mathfrak{R})$. Hence

$$TF(\mathfrak{R}) = CF(\mathfrak{R}) = F(\mathfrak{R})C = (C^*F(\mathfrak{R}))^* = (T^*F(\mathfrak{R}))^* = F(\mathfrak{R})T$$

as desired. □

COROLLARY 1. *If $T \in \theta$ is completely nonnormal, then $F(\mathfrak{R}) = 0$, or equivalently, $C - C^*$ is one-to-one.*

COROLLARY 2. *If $T \in \theta$ and $\sigma(T) \subseteq \mathfrak{R}$, then $T = T^*$.*

Corollary 2 follows from Proposition 1 and Theorem 1.

In [8] (see also [4]) it was shown how to get a block decomposition for $T \in \theta$ if $(T^*T - TT^*)$ was not one-to-one. For an arbitrary T, $[T^*, T]$ may be invertible. Whether $T \in \theta$ implies $[T^*, T]$ has a kernel is unknown. Note, however, that

PROPOSITION 3. *If $T \in \theta$, then $0 \in \sigma([T^*, T])$.*

PROOF. We may assume T is nonnormal. Then $\sigma(T) \nsubseteq \mathfrak{R}$ by Corollary 2. Hence there exists λ_0 in the approximate point spectrum of T, λ_0 not real. Thus there exists ϕ_n, $\|\phi_n\| = 1$, such that $(T - \lambda_0)\phi_n \to 0$. Then $B(\lambda_0)\phi_n \to 0$. But $B(\lambda_0)$ is normal, so that $B(\lambda_0)^*\phi_n = B(\bar{\lambda}_0)\phi_n \to 0$. Since

$$(\bar{\lambda}_0 - T^*)(\bar{\lambda}_0 - T)\phi_n = (\bar{\lambda}_0 - \lambda_0)(\bar{\lambda}_0 - T^*)\phi_n + (\bar{\lambda}_0 - T^*)(\lambda_0 - T)\phi_n,$$

we have $(T^* - \bar{\lambda}_0)\phi_n \to 0$ also. Now $[T^*, T] = [T^* - \bar{\lambda}_0, T - \lambda_0]$. Thus $[T^*, T]\phi_n \to 0$ and $0 \in \sigma([T^*, T])$. □

Let $r(T)$ denote the spectral radius of T.

THEOREM 2. *If $T \in \theta$, then $r(T) = \|T\|$. That is, T is normaloid.*

PROOF.

$$r(T)^2 = \sup_{\lambda \in \sigma(T)} |\lambda|^2 = \sup_{\lambda \in \sigma(T) \cup \sigma(T^*)} |\lambda|^2$$

$$= \sup_{\lambda \in \partial\sigma(T) \cup \partial\sigma(T^*)} |\lambda|^2 = \sup_{\lambda \in \hat{\sigma}(B)} |\lambda|^2 = \sup_{\lambda \in \sigma(C) \cup \sigma(C^*)} |\lambda|^2$$

$$= \|C^* C\| = \|T^* T\| = \|T\|^2. \quad \square$$

4. $\sigma(T) = \sigma(T^*)$. If $T = A + Q$ where $A = A^*$, $[A, Q] = 0$, and $[Q, Q^* Q] = 0$, then $T \in \theta$ and $\sigma(T)$ is the union of discs centered on the real axis. That such T are in θ is obvious. That $\sigma(T)$ is a union of discs follows from the canonical form for operators Q such that $[Q, Q^* Q] = 0$ given in [3] and the fact that the spectrum of the unilateral shift is a disc [7]. The results of this and the next section show that the spectrum of any $T \in \theta$ has many of the same features as a union of discs.

THEOREM 3. *If $T \in \theta$ and T is completely nonnormal, then $\sigma(T) = \sigma(T^*)$.*

PROOF. Suppose $T \in \theta$. It suffices to show that $\sigma(T) \subseteq \sigma(T^*)$. Note that $\sigma(T) \backslash \sigma(T^*) \subseteq \sigma(C) \cup \sigma(C^*)$. Hence, if K is any compact subset of $\sigma(T) \backslash \sigma(T^*)$ containing a set relatively open in $\sigma(T) \backslash \sigma(T^*)$, then $F(K) \neq 0$. Note also that $K \cap \mathfrak{R} = \varnothing$. Assume $\sigma(T) \not\subseteq \sigma(T^*)$. There exists, then, a compact set $K \subset \sigma(T) \backslash \sigma(T^*)$, $F(K) \neq 0$, and a Jordan contour Ω around K such that $\sigma(T^*)$ is contained in the unbounded component of the complement of Ω. Let $\tilde{C} = CF(K)$, $\tilde{B}(\lambda) = (\lambda - \tilde{C})(\lambda - \tilde{C}^*)$. Assume K is in the upper half plane. A similar proof works if K is in the lower half plane. Note that $\hat{\sigma}(\tilde{B}) = K \cup \overline{K}$ and $B(\lambda) F(K) = \tilde{B}(\lambda) F(K)$. Now for $\lambda \in \Omega$,

$$(\lambda - T^*)^{-1} F(K) = (\lambda - T^*)^{-1} \tilde{B}(\lambda) \tilde{B}^{-1}(\lambda) F(K)$$
$$= (\lambda - T^*)^{-1} B(\lambda) \tilde{B}^{-1}(\lambda) F(K) = (\lambda - T) \tilde{B}^{-1}(\lambda) F(K).$$

But $\int_\Omega (\lambda - T^*)^{-1} d\lambda = 0$. Thus

$$0 = \int_\Omega (\lambda - T) \tilde{B}^{-1}(\lambda)(\tilde{C} - \tilde{C}^*) F(K)\, d\lambda$$
$$= \int_\Omega (\lambda - T)\{(\lambda - \tilde{C})^{-1} - (\lambda - \tilde{C}^*)^{-1}\} F(K)\, d\lambda$$
$$= \int_\Omega (\lambda - T)(\lambda - \tilde{C})^{-1} F(K)\, d\lambda$$
$$= (\tilde{C} - T) F(K) = (C - T) F(K).$$

But $C + C^* = T + T^*$ so that we have

$$TF(K) = CF(K) = F(K)C = (C^* F(K))^* = (T^* F(K))^* = F(K)T.$$

Hence $F(K)$ reduces T and $TF(K)$ is normal which contradicts the complete nonnormality of T. $\square$

COROLLARY 3. *If $T \in \theta$, then $T = T_1 \oplus T_2$ where $T_1 \in \theta$, T_1 is completely nonnormal, $\sigma(T_1) = \sigma(T_1^*)$, and T_2 is normal.*

5. **Reducing components.** We shall say that a set of complex numbers S is *balanced* if: $\lambda \in S$ if and only if $\bar{\lambda} \in S$. A subset of $\sigma(T)$ will be called a *piece* if it is both open and closed in the topology induced on $\sigma(T)$ by the complex numbers.

THEOREM 4. *If $T \in \theta$ and K is a balanced piece of $\sigma(T)$, then relative to the decomposition of $\mathcal{H}$ given by $F(K)$, $T = T_1 \oplus T_2$ where $\sigma(T_1) = K$ and $\sigma(T_2) = \sigma(T)\backslash K$.*

PROOF. Take a balanced piece K of $\sigma(T)$. Note that $K \cap \hat{\sigma}(B)$ is a balanced piece of $\hat{\sigma}(B)$. Let Ω be a (possibly disconnected) contour around K with $\sigma(T)\backslash K$ on the outside. Assume for simplicity that T is completely nonnormal.
Now

$$\int_\Omega (C - C^*)B(\lambda)^{-1}\,d\lambda = \int_\Omega (\lambda - C)^{-1} - (\lambda - C^*)^{-1}\,d\lambda$$
$$= F_C(K) - F_{C^*}(K) = F_C(K) - F_C(K) = 0.$$

But $C - C^*$ is one-to-one by Corollary 1 so that

$$\int_\Omega B(\lambda)^{-1}\,d\lambda = 0. \tag{2}$$

We also have that

$$\int_\Omega \lambda(C - C^*)B(\lambda)^{-1}\,d\lambda = \int_\Omega \lambda(\lambda - C)^{-1} - \lambda(\lambda - C^*)^{-1}\,d\lambda$$
$$= (C - C^*)F(K)$$

so that

$$\int_\Omega \lambda B(\lambda)^{-1}\,d\lambda = F(K). \tag{3}$$

But then by (2) and (3) we have

$$\int_\Omega (\lambda - T)^{-1}\,d\lambda = \int_\Omega B(\lambda)^{-1}(\lambda - T^*)\,d\lambda = \int_\Omega \lambda B(\lambda)^{-1}\,d\lambda = F(K).$$

Thus $F(K)$ reduces T and $\sigma(T|F(K)\mathcal{H}) = K$ as desired. □

PROPOSITION 4. *If $T \in \theta$ and $\sigma(T)$ is a subset of a vertical line, then T is normal.*

PROOF. Let x_0 be real and suppose $T \in \theta$, $\sigma(T) \subseteq \{\lambda: \operatorname{Re}\lambda = x_0\}$. Then $T^* + T = C^* + C = 2x_0 I$ by Proposition 1 and the fact that $\hat{\sigma}(B) = \sigma(C) \cup \sigma(C^*)$. Hence, $[T, T^*] = 0$ and T is normal. □

THEOREM 5. *If λ_0 is an isolated point of $\sigma(T)$ and $T \in \theta$, then λ_0 is a reducing eigenvalue of T.*

PROOF. We may assume that T is completely nonnormal and hence λ_0, $\bar{\lambda}_0$ are both isolated. Thus $\{\lambda_0, \bar{\lambda}_0\}$ is a balanced piece of $\sigma(T)$. That λ_0 is a reducing eigenvalue now follows from Theorem 4 and Proposition 4. □

6. **Comment.** Using the terminology of [1] and [2] we have shown that if $T \in \theta$, then T is reduction-normaloid, reduction-spectraloid, spectraloid, isoloid, and reduction-isoloid.

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