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FIXED POINTS AND PARTIAL ORDERS

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ABSTRACT. It is observed that certain fixed point theorems may be derived from theorems on the existence of maximal elements in partially ordered sets.

The main point of the present note is to show how certain fixed point theorems can be deduced from the following simple observation:

(A) *Let* $(E, \leqslant)$ *be a partially ordered set which admits at least one maximal element. Let* T *be a mapping from* E *into* E *such that* $s \leqslant Tx$ *for all* $x \in E$. *Then* T *admits at least one fixed point.*

Among the partially ordered sets having maximal elements we shall call attention to a certain type first considered by E. Bishop and R. R. Phelps. Let (E, d) be a metric space, and let φ be a real valued function on E. Define $\leqslant_{d,\varphi}$ by letting $x \leqslant_{d,\varphi} y$ if and only if $d(x,y) \leqslant \varphi(x) - \varphi(y)$. Then $\leqslant_{d,\varphi}$ is a partial order, which admits at least one maximal element provided that (E, d) is complete and φ is lower semicontinuous and bounded below. (For details, see Theorem 2 of [1]. Actually, this theorem is more general than just indicated.) Therefore, invoking also (A) we obtain:

(B) *Let* (E, d) *be a complete metric space and let* φ *be a real valued function on* E *which is lower semicontinuous and bounded below. Let* T *be a mapping from* E *into* E *such that* $d(x, Tx) \leqslant \varphi(x) - \varphi(Tx)$ *for each* $x \in E$. *Then* T *admits at least one fixed point.*

This result is stated as the main result of [2] (with no indication of its relationship to the ideas described above). It is used to prove fixed point theorems for so-called inward mappings in Banach spaces. To illustrate how (A) may be used to establish other types of fixed point theorems, we shall state the following:

(C) *Let* F *be a closed subset of a Banach space* X, *let* $z \in X \setminus F$, *and let* $0 < r < \inf_{y \in F}\|z - y\|$. *Let* T: $F \to F$ *be a mapping which maps each* $x \in F$

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in the direction of the closed ball $B(z, r)$, *i.e. if* $Tx \neq x$, *then* $x + t(Tx - x) \in B(z, r)$ *for some* $t > 1$. *Then* T *has at least one fixed point.*

In fact, define $\leqslant$ on F by letting $x \leqslant y$ if and only if

$$x = y \quad \text{or} \quad x + t(y - x) \in B(z, r) \quad \text{for some } t > 1.$$

Then $\leqslant$ is a partial order on F which is finer than a certain order of the type $\leqslant_{d,\varphi}$. Since $(F, \leqslant_{d,\varphi})$ admits at least one maximal element by Theorem 2 of [1], it follows that $(F, \leqslant)$ admits a maximal element. (For details, see the proof of Theorem 3 of [1].) Application of (A) then yields (C).

References

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