EIGENVALUES OF HOPF MANIFOLDS
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Abstract. The eigenvalues of the Laplacians $\Delta$ and $\Box$ on the Hopf manifolds are described. Some isospectral results are also given.

On a complex manifold $M$ there are the de Rham and Dolbeault complexes, with operators $d$ and $\bar{\partial}$, respectively. If we fix a hermitian metric on the manifold $M$, then $d$ and $\partial$ will have formal adjoints $\delta$ and $\partial$ with respect to the hermitian volume element. With these, we define the real Laplacian $\Delta = d\delta + \delta d$ and the complex Laplacian $\Box = \delta \partial + \partial \delta$. We have $\Delta = 2 \Box = 2 \partial \partial$ if $M$ is Kähler.

In this note, we study the eigenvalues of the Laplacians $\Delta$ and $\Box$ on the Hopf manifolds $M_\alpha$, $0 < |\alpha| < 1$, which are not Kähler. The notation $\text{Sp}(M, \Delta)$ will be used to denote the eigenvalues with multiplicity (spectrum) of the operator $\Delta$ on the manifold $M$. In Theorem 1, the eigenvalues of $\Delta$ and $\Box$ are described. The computation of the eigenfunctions shows that the eigenspaces of $\Box$ refine those of $\Delta$. Some isospectral results are also given. For $\alpha$ in a certain domain, $\text{Sp}(M_\alpha, \Delta)$ determines $M_\alpha$ up to isometry (Theorem 2). Moreover, it is shown that for “most” values of $\alpha$, $M_\alpha$ may be determined up to isometry from either $\text{Sp}(M_\alpha, \Delta)$ or $\text{Sp}(M_\alpha, \Box)$ (Theorem 3).

Let $W$ denote the $n$-dimensional complex vector space $\mathbb{C}^n = \{z | z = (z_1, \ldots, z_n)\}$ minus the origin; $W = \mathbb{C}^n - \{0\}$, and let $\alpha$ be a complex number with $0 < |\alpha| < 1$. Consider the analytic automorphism $g_\alpha$ of $W$ defined by $g_\alpha(z_1, \ldots, z_n) = (\alpha z_1, \ldots, \alpha z_n)$. The group $G_\alpha$ generated by $g_\alpha$ is an infinite cyclic group acting on $W$ freely and properly discontinuously. Thus the quotient $M_\alpha = W/G_\alpha$ is an $n$-dimensional complex manifold, which is called a (homogeneous) Hopf manifold. It is easy to see that $M_\alpha$ is diffeomorphic to $S^1 \times S^{2n-1}$, where $S^r$ denotes the standard $r$-sphere (cf. the proof of Lemma 3). If $n = 1$, $M_\alpha$ is the complex torus whose lattice is generated by 1 and $(2\pi i)^{-1}\log \alpha$, and if $n > 1$, $M_\alpha$ is a non-Kähler manifold. The hermitian metric

\begin{equation}
 g = ||z||^{-2} \sum_{i=1}^{n} dz_i d\bar{z}_i, \quad ||z||^2 = \sum_{i=1}^{n} z_i \bar{z}_i,
\end{equation}

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on $W$ is $G_a$-invariant. Hence, it induces a hermitian metric on $M_a$. From now on we think of $M_a$ as a hermitian manifold with this metric. It is easy to check that if $n = 1$, $M_a$ is a flat torus. If $a$ is real, $M_a$ is isometric to $S^1 \times S^{2n-1}$ as a riemannian manifold. We denote by $\mathcal{C}^\infty(M_a)$ the space of complex valued smooth functions on $M_a$. Consider the real and complex Laplacians $\Delta$ and $\Box$ induced by the hermitian metric on $M_a$. Since the operator $\bar{\partial}$ maps a $(p,q)$-form to a $(p,q - 1)$-form, we have

\[
\Delta = \Box + \Box \quad \text{on } \mathcal{C}^\infty(M_a).
\]

A straightforward computation [3, p. 97] shows that the complex Laplacian on $\mathcal{C}^\infty(M_a)$ is

\[
\Box = -\|z\|^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} + (n - 1) \sum_{i=1}^{n} \bar{z}_i \frac{\partial}{\partial z_i}.
\]

Let

\[
\Delta_0 = - \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}
\]

be the standard Laplacian on $\mathbb{C}^n$. Moreover, let $\mathcal{H}_{p,q}$ be the space of harmonic polynomials of type $(p,q)$, i.e., the polynomials $f$ on $\mathbb{C}^n$ such that $\Delta_0 f = 0$ and $f(z) = \sum_{|\mu|=p; |\nu|=q} c_{\mu \nu} z^\mu z^{\nu}$, where $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ are $n$-tuples of nonnegative integers, $|\mu| = \sum_{i=1}^{n} \mu_i$, $|\nu| = \sum_{i=1}^{n} \nu_i$, and $z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$, $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$.

**Lemma 1.** For $f \in \mathcal{H}_{p,q}$ and for a complex number $\gamma$,

\[
\Box (\|z\|^{\gamma} f) = \left(-\frac{\gamma}{2}\right)^2 \left(\|z\|^2 f - \frac{\gamma}{2} \|z\|^{\gamma} \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}\right) + \left(n - 1 - \frac{\gamma}{2}\right) \|z\|^{\gamma} \sum_{i=1}^{n} z_i \frac{\partial f}{\partial \bar{z}_i}.
\]

**Proof.** Since $\Delta_0 f = 0$, we have

\[
\Box (\|z\|^{\gamma} f) = \left(-\frac{\gamma}{2}\right)^2 \|z\|^2 f - \frac{\gamma}{2} \|z\|^\gamma \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} + \left(n - 1 - \frac{\gamma}{2}\right) \|z\|^\gamma \sum_{i=1}^{n} z_i \frac{\partial f}{\partial \bar{z}_i}.
\]

Substituting

\[
\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = pf \quad \text{and} \quad \sum_{i=1}^{n} z_i \frac{\partial f}{\partial \bar{z}_i} = qf
\]

in the above equation, we get (3). Q.E.D.

Let $\omega$ be a complex number such that $e^{2\pi i \omega} = \alpha$. Since $|\alpha| < 1$, we have $\text{Im} \omega > 0$.

**Definition.** For nonnegative integers $p$ and $q$, let $\Gamma_{p,q}$ be the set of complex numbers $\gamma$ which can be expressed as $\gamma = -\gamma_1 + \gamma_2$ with $\text{Re} \gamma_1 = p$, $\text{Re} \gamma_2 = -q$, and $\text{Re}((\gamma_1 + \gamma_2)\omega) \in \mathbb{Z}$.

**Remarks.** 1°. $\gamma_1 + \gamma_2$ is in the lattice dual to the one generated by $1$ and $\omega$ (cf. [1, p. 146]).
2°. Let \( a = \text{Re} \omega \) and \( b = \text{Im} \omega \). Then

\[
\Gamma_{p,q} = \{ -(p + q) + ((a(p - q) - k)/b)i | k \in \mathbb{Z} \}.
\]

**Lemma 2.** For \( f \in \mathcal{K}_{p,q} \), the function \( \|z\|^{\gamma}f \) is \( G_{a}\)-invariant if and only if \( \gamma \in \Gamma_{p,q} \).

**Proof.** \( \|z\|^{\gamma}f \) is \( G_{a}\)-invariant if and only if \( |\alpha|^{\gamma}a^p\alpha^q = 1 \), or equivalently,

\[
\begin{aligned}
\text{Re} \gamma &= -(p + q) \\
-\text{Im} \omega \cdot \text{Im} \gamma + (p - q)\text{Re} \omega &= k, \quad \text{for some } k \in \mathbb{Z}.
\end{aligned}
\]

In view of (4), this is equivalent to \( \gamma \in \Gamma_{p,q} \). Q.E.D.

**Lemma 3.** The vector subspace of \( \mathcal{C}^{\infty}(M_a) \) generated by \( \mathcal{U}_{p,q \geq 0}(\|z\|^{\gamma}f | f \in \mathcal{K}_{p,q}, \gamma \in \Gamma_{p,q}) \) is dense in \( \mathcal{C}^{\infty}(M_a) \).

**Proof.** Consider the unit \((2n - 1)\)-sphere \( S^{2n-1} = \{ z \in \mathbb{C}^{n} | \|z\| = 1 \} \). The map \( \varphi: \mathbb{R} \times S^{2n-1} \rightarrow W \) defined by \( \varphi(t,z_1,\ldots,z_n) = (e^{2\pi it}z_1,\ldots,e^{2\pi it}z_n) \) is clearly a diffeomorphism. Moreover, \( \varphi \) induces a diffeomorphism, which we denote also by \( \varphi \), from \( S^1 \times S^{2n-1} \) onto \( M_a \), where \( S^1 = \mathbb{R}/\mathbb{Z} \). Take a function \( \|z\|^{\gamma}f \) with \( f \in \mathcal{K}_{p,q} \) and \( \gamma \in \Gamma_{p,q} \). The pullback of \( \|z\|^{\gamma}f \) by \( \varphi \) is given by

\[
((\|z\|^{\gamma}f) \circ \varphi)(t,z) = \exp(\frac{2\pi it}{\text{Im} \omega + p\omega - q\omega})f(z) = e^{2\pi ikt}f(z),
\]

where \( k = \text{Re}((\gamma_1 + \gamma_2)\alpha) \in \mathbb{Z} \). On the other hand, the vector subspace of \( \mathcal{C}^{\infty}(S^1) \) generated by the functions \( \{e^{2\pi ik}\} \) is dense in \( \mathcal{C}^{\infty}(S^1) \) and the harmonic polynomials are dense in \( \mathcal{C}^{\infty}(S^{2n-1}) \) [1, p. 160]. Q.E.D.

**Theorem 1.** (i) The eigenvalues of \( \square \) on \( \mathcal{C}^{\infty}(M_a) \) are \( |\gamma|^2/4 + q(n - 1), p, q \in \mathbb{Z}^+, \gamma \in \Gamma_{p,q} \).

(ii) The eigenvalues of \( \Delta \) on \( \mathcal{C}^{\infty}(M_a) \) are \( |\gamma|^2/2 + (p + q)(n - 1), p, q \in \mathbb{Z}^+, \gamma \in \Gamma_{p,q} \), where \( \mathbb{Z}^+ \) denotes the nonnegative integers.

**Proof.** If \( \gamma \in \Gamma_{p,q} \), we have \(-|\gamma|^2/2 - (p + q)|\gamma|/2 + q(n - 1) = |\gamma|^2/4 + q(n - 1)\). The theorem follows from (2), Lemmas 1–3, and Lemma A.II.1 in [1, p. 143]. Q.E.D.

Now let us discuss the isospectral problem. Let \( a + bi = \omega = (2\pi i)^{-1}\log \alpha \) as before. Since \( a = (2\pi)^{-1}\arg \alpha \), we may assume that \( |a| \leq \frac{1}{2} \). Note that if \( |a| = \frac{1}{2} \), then \( \alpha \) is real and \( M_a \) is isometric to \( S^1 \times S^{2n-1} \).

**Theorem 2.** Suppose (i) \( b > 1/2\sqrt{2n + 1} \) or (ii) \( |a| < b\sqrt{2n + 1} \). Then \( \text{Sp } (M_a, \Delta) \) determines \( M_a \) up to isometry, namely, \( \text{Sp } (M_a, \Delta) = \text{Sp } (M_{a'}, \Delta) \) implies \( M_a \) is isometric to \( M_{a'} \).

**Proof.** Let

\[
a' + b'i = \omega' = (2\pi i)^{-1}\log \alpha', \quad |a'| \leq \frac{1}{2}.
\]

If \( \text{Sp } (M_a, \Delta) = \text{Sp } (M_{a'}, \Delta) \), then the volumes of \( M_a \) and \( M_{a'} \) are the same [1, Corollaire E.IV.2, p. 216], see also [4]. This implies \( |a| = |a'| \) or equivalently \( b = b' \). Consider the eigenvalues
\[ \lambda(p, q, k, a^{(p)}) = \left( p + q \right)^2/2 + \left( a^{(p)}(p - q) - k \right)^2/2b^2 + (p + q)(n - 1), \]
p, q \in \mathbb{Z}^+, k \in \mathbb{Z}, of A on \mathbb{E}_\infty(M_{a^{(p)}}), \nu = 0, 1, a^{(0)} = a, a^{(1)} = a'. \text{ Note that if } p + q > 2, \text{ then } \lambda(p, q, k, a^{(p)}) > 2n \text{ and that } \lambda(p, q, k, a^{(p)}) = 0 \text{ if and only if } p = q = k = 0.

Case (i). \( b > \sqrt{2n + 1} \). Since \(|a| < b\), we have
\[
\lambda(1, 0, 0, a) = a^2/2b^2 + n - \frac{1}{2} < 2a^2(2n + 1) + n - \frac{1}{2} \leq \frac{1}{2}(2n + 1) + n - \frac{1}{2} = 2n.
\]
Similarly, we have \( \lambda(1, 0, 0, a') < 2n \). Hence, we have
\[
\lambda(1, 0, 0, a') = \lambda(p_1, q_1, k_1, a'), \quad \lambda(1, 0, 0, a) = \lambda(p_2, q_2, k_2, a)
\]
for some \( p_j, q_j \in \mathbb{Z}^+ \) and \( k_j \in \mathbb{Z} \) with \( 0 \leq p_j + q_j \leq 1, j = 1, 2 \). If \( p_1 + q_1 = 1 \), we have \( a^2 = (a' + k_1)^2 \). Since \(|a' + k_1| \leq \frac{1}{2} \), we get \( a' = \pm a \). If \( p_2 + q_2 = 1 \), we also get \( a' = \pm a \). If \( p_j + q_j = 0, j = 1, 2 \), we have \( a^2 - a'^2 = k_1^2 - k_2^2 \). Since \(|a^2 - a'^2| \leq \frac{1}{2} \), we get \( a' = \pm a \).

Case (ii). \( 0 < b \leq 1/2\sqrt{2n + 1} \). We have
\[
\lambda(1, 0, 0, a') = -\frac{k^2}{2b^2} > 2\sqrt{2n + 1} > 2n, \text{ for } |k| > 1.
\]
Hence, we have \( \lambda(1, 0, 0, a) = \lambda(p_1, q_1, k_1, a'), \) with \( p_1 + q_1 = 1 \). Thus, we get \( a' = \pm a \).

If \( a' = -a \), then \( a' = \bar{a} \) and the diffeomorphism \( W \to W \) defined by \( (z_1, \ldots, z_n) \mapsto (\bar{z}_1, \ldots, \bar{z}_n) \) induces an isometry between \( M_a \) and \( M_{a'} \). Q.E.D.

**Theorem 3.** Suppose \( a \) and \( b \) are algebraically independent over \( \mathbb{Q} \). Then \( M_a \) may be determined up to isometry from either \( \text{Sp} (M_a, \Delta) \) or \( \text{Sp} (M_a, \Box) \).

**Proof.** We show that \( \text{Sp} (M_a, \Box) \) determines \(|a|\) and \( b \). The same argument with only small modifications will apply to \( \text{Sp} (M_{a'}, \Delta) \). Let \( a' \) be a complex number with \( a' + ib' = (2\pi)^{-1}\log a', -\frac{1}{2} < a' \leq \frac{1}{2} \). We shall assume that \( \text{Sp}(M_{a'}, \Box) = \text{Sp}(M_a, \Box) \) and show that \(|a| = |a'| \) and \( b = b' \). Since the volume of \( M_a \) determines \( b \), and the volume of \( M_a \) is determined by the asymptotic behavior of the spectrum (see, for instance, Gilkey [4]), it follows that \( b = b' \). Let
\[
\lambda(p, q, k, a, b) = (p + q)^2/4 + (a(p - q) - k)^2/4b^2 + (n - 1)q.
\]
It is easily seen that the linear span of \( \{b^2\lambda | \lambda \in \text{Sp} (M_a, \Box) \} \) over \( \mathbb{Q} \) is the linear span over \( \mathbb{Q} \) of \( \{1, a, a^2, b^2\} \). Thus if \( \text{Sp} (M_a, \Box) = \text{Sp} (M_{a'}, \Box) \), then \( \{1, a, a^2, b^2\} \) and \( \{a', (a')^2, b^2\} \) have the same linear span over \( \mathbb{Q} \). Thus there exist rationals \( r_j, r'_j, 1 \leq j \leq 4 \), such that
\[
a' = r_1 + r_2a + a^2r_3 + b^2r_4, \quad (a')^2 = r_1' + r_2a' + a^2r_3' + b^2r_4'.
\]
By the algebraic independence of \(a\) and \(b\), it follows by squaring the left-hand equation that \(r_3 = r_4 = 0\). Thus \(a' = r_1 + ar_2\).

Now we show that \(r_1 = 0\), \(r_2 = \pm 1\). Observe that if

\[
\text{Sp} (M_\alpha, \square) = \text{Sp} (M_{\alpha'}, \square)
\]

then for integers \((p, q, k)\) there exists \((p', q', k')\) such that

\[
\lambda(p, q, k, a, b) = \lambda(p', q', k', r_1 + ar_2, b).
\]

Conversely, for \((p', q', k')\) there exists \((p, q, k)\). Multiplying (5) by \(b^2\) and comparing the coefficients of 1 and \(a^2\), we obtain

\[
k^2 = ((p' - q')r_1 - k')^2,
\]

\[
(p - q)^2 = ((p' - q')r_2)^2.
\]

Since (7) must always have integer solutions, \(r_2 = \pm 1\). From equation (6) it follows that \(r_1\) must be an integer, and so \(r_1 = 0\) since \(-\frac{1}{2} \leq a' \leq \frac{1}{2}\). Thus \(|a|\) and \(b\) are determined. Q.E.D.

Remarks. 1. If \(n = 1\), \(M_\alpha\) and \(M_{\alpha'}\) are biholomorphic, as is well known, if and only if \((\zeta') = u(\zeta)\) for some \(u \in \text{SL}(2, \mathbb{Z})\), where

\[
\omega = (2\pi i)^{-1} \log \alpha \quad \text{and} \quad \omega' = (2\pi i)^{-1} \log \alpha'.
\]

If \(n \geq 2\), \(M_\alpha\) and \(M_{\alpha'}\) are biholomorphic, by Hartogs' theorem, if and only if \(\alpha' = \alpha\) [2, Theorem 15.1].

2. It does not seem easy to find the dimension (multiplicity) of each eigenspace. The dimension of \(\mathcal{E}_{p,q}\) can be computed as follows. Let \(\mathcal{P}_{p,q}\) denote the space of polynomials on \(\mathbb{C}^d\) of type \((p, q)\) and set \(r = \|z\|\). Observe that \(\mathcal{E}_k\), the harmonic polynomials homogeneous of degree \(k\), is subdivided into the spaces \(\mathcal{E}_{p,q}\), i.e., \(\mathcal{E}_k = \bigoplus_{p+q=k} \mathcal{E}_{p,q}\). This may be done since \(\Delta_0\) maps \(\mathcal{P}_{p,q}\) into \(\mathcal{P}_{p+1,q-1}\). By repeating the argument in Berger [1, p. 161], we may conclude that for \(p \geq q\),

\[
\mathcal{P}_{p,q} = \mathcal{E}_{p,q} \oplus r^2 \mathcal{E}_{p-1,q-1} \oplus \cdots \oplus r^{2q} \mathcal{E}_{p-q,0},
\]

and the summands are pairwise orthogonal in \(L^2(S^{2n-1})\). Thus, \(\mathcal{P}_{p,q} = \mathcal{E}_{p,q} \oplus r^2 \mathcal{P}_{p-1,q-1}\) from which it follows that

\[
\dim \mathcal{E}_{p,q} = \dim \mathcal{P}_{p,q} - \dim \mathcal{P}_{p-1,q-1}.
\]

Since

\[
\dim \mathcal{P}_{p,q} = \binom{n-1+p}{p} \binom{n-1+q}{q},
\]

we get

\[
\dim \mathcal{E}_{p,q} = \binom{n+p-1}{p} \binom{n+q-1}{q} - \binom{n+p-2}{p-1} \binom{n+q-2}{q-1}.
\]
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