A CHARACTERIZATION OF \( \mu \)-SEMIRINGS

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Abstract. A characterization of \( \mu \)-semirings is given, namely, "A semiring \( \mathcal{S} \) is a \( \mu \)-semiring, if and only if, for each ideal \( a \) of \( \mathcal{S} \) with no subideals in a \( \pi \)-system \( \mathcal{B} \), there exists a maximal ideal which has no subideals in \( \mathcal{B} \) and contains \( a \)."

1. Introduction. A semiring is an algebraic system \( \mathcal{S} = \{a, b, c, \ldots \} \) in which two binary associative operations, called sum (\( + \)) and product (\( \cdot \)), are defined so that the operation \( \cdot \) is both left- and right-distributive over \( + \). A subset \( a \) of \( \mathcal{S} \) is called an ideal if: (i) \( a, b \in a \) imply \( a + b \in a \); (ii) \( a \in a, s \in \mathcal{S} \) imply \( as \in a, sa \in a \).

A subset \( M \) of \( \mathcal{S} \) is called an \( m \)-system of \( \mathcal{S} \) if, for each pair \( a, b \in M \), there exists \( x \in \mathcal{S} \) such that \( axb \in M \); a subset \( P \) of \( \mathcal{S} \) is called a \( p \)-system of \( \mathcal{S} \) if, for each \( a \in P \), there exists \( x \in P \) such that \( axa \in P \). These concepts, stemming from ring theory, allow us, as in that theory, to make the study of prime and semiprime ideals and to introduce the notion of the Baer-McCoy-Levitzki radical [1].

Lattice semirings are instances of interesting semirings. \( \mathcal{S} \) is a lattice semiring if: (i) \( \mathcal{S} \) is a lattice besides being a semiring; (ii) the operations \( \wedge, \vee \) satisfy \( x + y = x \vee y, xy \leq x \wedge y \). For these semirings, M. L. Noronha Galvão gave [5] a theory for primary and primal ideals analogous to the theory of Noether-Krull-Fuchs.

Important examples of lattice semirings are the sets \( \mathcal{S} \) of all ideals either of a ring or of a semiring or of a semigroup. \( m \)-systems and \( p \)-systems of \( \mathcal{S} \) are called by A. Almeida Costa [2], respectively, \( \mu \)-systems and \( \pi \)-systems of \( \mathcal{S} \). Consequently, leaving aside \( \mathcal{S} \), a set \( \mathcal{M} \) of ideals of a semiring \( \mathcal{S} \) is a \( \mu \)-system, if and only if, for each pair \( a, b \in \mathcal{M} \), there exists an ideal \( \mathcal{I} \) of \( \mathcal{S} \) such that \( axb \in \mathcal{I} \); a set \( \mathcal{P} \) of ideals of \( \mathcal{S} \) is a \( \pi \)-system if and only if, for each \( a \in \mathcal{P} \), there exists an ideal \( \mathcal{I} \) of \( \mathcal{S} \) such that \( axa \in \mathcal{I} \). Moreover, in any semiring \( \mathcal{S} \) the set of all ideals which are not contained in a given prime ideal is a \( \mu \)-system and the set of all ideals which are not contained in a given semiprime ideal is a \( \pi \)-system.

A \( \mu \)-semiring is a semiring which satisfies either the \( \mu \)-condition or the...
$\pi$-condition. These conditions are defined as follows (we denote by $C(\pi)$ the set of all ideals which are not subideals of $\pi$):

$(\mu)$ For every $\mu$-system $\mathcal{M}$ and every chain of ideals $\{a_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathcal{M} \subseteq C(a_\lambda)$ ($\lambda \in \Lambda$) one has $\mathcal{M} \subseteq C(\bigcup a_\lambda)$;

$(\pi)$ For every $\pi$-system $\mathcal{P}$ and every chain of ideals $\{a_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathcal{P} \subseteq C(a_\lambda)$ ($\lambda \in \Lambda$) one has $\mathcal{P} \subseteq C(\bigcup a_\lambda)$.

These assertions are equivalent, as proved in [4] and [5] where the theory of $\mu$-semirings is developed. These assertions are also equivalent to the following:

$(\mu_1)$ For every $\mu$-system $\mathcal{M}$ and every chain of ideals $\{a_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathcal{M} \subseteq C(a_\lambda)$ ($\lambda \in \Lambda$) there is an ideal $a$ such that $a_\lambda \subseteq a$ ($\lambda \in \Lambda$), $\mathcal{M} \subseteq C(a)$;

$(\pi_1)$ For every $\pi$-system $\mathcal{P}$ and every chain of ideals $\{a_\lambda\}$ ($\lambda \in \Lambda$) such that $\mathcal{P} \subseteq C(a_\lambda)$ ($\lambda \in \Lambda$) there is an ideal $a$ such that $a_\lambda \subseteq a$ ($\lambda \in \Lambda$), $\mathcal{P} \subseteq C(a)$.

Noetherian semirings, that is, those which satisfy the a.c.c. for ideals (in particular, semirings of finite order) and non-Noetherian semirings consisting of the real numbers $x > r$, where $r > 1$ is a real number [3], provide examples of $\mu$-semirings.

In the general theory of semirings the use of certain $\mu$-systems and certain $\pi$-systems (said "particulars") has permitted the establishment of results concerning prime and semiprime ideals and consequent radical theories, but in the theory of $\mu$-semirings the use of $\mu$-systems and $\pi$-systems is sufficient to establish the Noether-Krull-Fuchs results.

Let us take in a $\mu$-semiring a $\mu$-system $\mathcal{M}$ ($\pi$-system $\mathcal{P}$) and an ideal $a$ with no subideals in $\mathcal{M}$ (in $\mathcal{P}$). From Zorn's lemma it follows that there is a maximal ideal which has no subideals in $\mathcal{M}$ (in $\mathcal{P}$) and contains $a$.

In this note we will prove the following characterization of $\mu$-semirings:

A semiring $\mathcal{S}$ is a $\mu$-semiring if and only if it satisfies the condition:

$(\pi_0)$ For each ideal $a$ and for each $\pi$-system $\mathcal{P}$ such that $a$ has no subideals in $\mathcal{P}$, i.e., $\mathcal{P} \subseteq C(a)$, there exists a maximal ideal $\eta$ which has no subideals in $\mathcal{P}$ and contains $a$, i.e., $\mathcal{P} \subseteq C(\eta) \subseteq C(a)$.

2. Preliminary propositions. We first prove:

**Proposition 1.** Let $\mathcal{P}$ be a $\pi$-system. If there is a maximal ideal $\eta$ with no subideals in $\mathcal{P}$, i.e., $\mathcal{P} \subseteq C(\eta)$, then $\eta$ is a semiprime ideal.

**Proof.** Let us assume that $\eta$ is not semiprime. Then for an ideal $\pi$ one has $\pi^2 \subseteq \eta$, $\pi \not\subseteq \eta$. Hence $\eta \subset (\pi, \eta)$, the least ideal containing both $\pi$ and $\eta$. Since $\eta$ is maximal and has no subideals in $\mathcal{P}$, there exists $m \in \mathcal{P}$ such that $m \subseteq (\pi, \eta)$. Let us consider an ideal $\xi$ such that $m^2 \subseteq (\xi, \mathcal{M}) \subseteq \eta$ contradict the hypothesis about $\eta$. Hence $\pi^2 \subseteq \eta$ implies $\pi \subseteq \eta$.

Let $\mathcal{P}$ be a $\pi$-system and $a$ an ideal such that $\mathcal{P} \subseteq C(a)$; then a maximal ideal $\eta$ such that $\mathcal{P} \subseteq C(\eta) \subseteq C(a)$ is, of course, a maximal ideal satisfying $\mathcal{P} \subseteq C(\eta)$. We have:
**Corollary 1.** Let $\mathcal{B}$ be a $\pi$-system and $\alpha$ an ideal such that $C(\alpha)$. If there is a maximal ideal $\eta$ such that $\mathcal{B} \subseteq C(\eta) \subseteq C(\alpha)$, then $\eta$ is a semiprime ideal.

**Lemma 1.** Let $\alpha$ be a semiring satisfying condition $(\pi_0)$. Given a $\pi$-system $\mathcal{B}$ and an ideal $\alpha$ such that $\mathcal{B} \subseteq C(\alpha)$, then there exists a minimal semiprime ideal $\bar{\mathcal{B}}$ such that $\mathcal{B} \subseteq C(\bar{\mathcal{B}}) \subseteq C(\alpha)$.

**Proof.** Condition $(\pi_0)$ implies the existence of a maximal ideal $\eta$ such that $\mathcal{B} \subseteq C(\eta) \subseteq C(\alpha)$. Since, by Corollary 1, $\eta$ is semiprime, the intersection of all semiprime ideals $\eta$ such that $\mathcal{B} \subseteq C(\eta) \subseteq C(\alpha)$ is the minimal semiprime ideal $\bar{\mathcal{B}}$ we are looking for.

Now, let $\mathfrak{g}$ be a family of ideals of a semiring $\mathfrak{S}$ satisfying the following conditions: $(G_1)$ $g_1, g_2 \in \mathfrak{g}$ imply $(g_1, g_2) \in \mathfrak{g}$; $(G_2)$ $g \subseteq g_1 \in \mathfrak{g}$ imply $g \in \mathfrak{g}$ ($\mathfrak{g}$ is an ideal of the lattice $\mathfrak{S}$ of all ideals of $\mathfrak{S}$). It is easy to verify that the existence of a maximal element $g_0 \in \mathfrak{g}$ implies $g_0 = \bigcup g (g \in \mathfrak{g})$. It is the same to say that $g_0$ is maximal in $\mathfrak{g}$ or to say that $g_0$ is maximal such that $\mathfrak{g} - g = C(\eta_0)$.

**Lemma 2.** Let $\mathfrak{S}$ be a semiring satisfying condition $(\pi_0)$ and let $\{S_\lambda\} (\lambda \in \Lambda)$ be a chain of semiprime ideals; then $\bigcup S_\lambda = S_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

**Proof.** Let $\mathfrak{g}$ be the family consisting of all subideals of all $S_\lambda$. $\mathfrak{g}$ satisfies $(G_1)$ and $(G_2)$. We shall verify that the set of all ideals not in $\mathfrak{g}$, $\mathcal{B} = \mathfrak{S} - \mathfrak{g}$, is a $\pi$-system. Given $\mathfrak{x} \in \mathcal{B}$ we shall prove that $\mathfrak{g} \mathfrak{x} \mathfrak{g} \in \mathcal{B}$. If this were not so, one would have $\mathfrak{g} \mathfrak{x} \mathfrak{g} \subseteq \mathfrak{g}$, hence $\mathfrak{g} \mathfrak{x} \mathfrak{g} \subseteq S_\lambda$, for some $\lambda \in \Lambda$, which would imply $\mathfrak{x} \subseteq S_\lambda$, i.e., $\mathfrak{x} \in \mathfrak{g}$, which is absurd. The fact that $\mathfrak{S}$ satisfies condition $(\pi_0)$ and the inclusion $\mathcal{B} \subseteq C(S_\lambda)$, together, imply the existence of a maximal ideal $\eta$ such that $\mathcal{B} \subseteq C(\eta)$. Thus we conclude the existence of a maximal ideal in $\mathfrak{g}$, which is necessarily a $S_\lambda$ such that $\bigcup S_\lambda = S_{\lambda_0} (\lambda \in \Lambda)$.

3. **Main proposition.** We have seen above, in the introduction, that the necessity of condition $(\pi_0)$ for $\mathfrak{S}$ to be a $\mu$-semiring is a consequence of Zorn’s lemma. Conversely, let $\mathfrak{S}$ be a semiring that satisfies condition $(\pi_0)$, let $\mathcal{B}$ be a $\pi$-system, and let $\{a_\lambda\} (\lambda \in \Lambda)$ be a chain of ideals of $\mathfrak{S}$ such that $\mathcal{B} \subseteq C(a_\lambda)$ (H = $\Lambda$). By Lemma 1, we can assign to each $a_\lambda$ the minimal semiprime ideal $S_\lambda$ such that $\mathcal{B} \subseteq C(S_\lambda) \subseteq C(a_\lambda)$. From $a_\alpha \subseteq a_\lambda$, one concludes $\mathcal{B} \subseteq C(S_\lambda) \subseteq C(a_\lambda) \subseteq C(a_\alpha)$, hence by the minimality of $S_\alpha$, $S_\alpha \subseteq S_\lambda$. Then, by Lemma 2 and by the fact that $a_\lambda \subseteq S_\lambda$, $a_\alpha \subseteq S_\alpha$, we have $S_\lambda = S_{\lambda_0}$; consequently, $\mathcal{B} \subseteq C(S_{\lambda_0}) \subseteq C(\bigcup a_\lambda)$. This completes the proof of the main proposition.

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**Bibliography**


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