VANISHING SOLUTIONS OF THE DISSIPATIVE ACOUSTIC EQUATION IN AN EXTERIOR DOMAIN

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ABSTRACT. Except in one dimension, strictly incoming waves cannot be propagated by the wave equation with dissipative boundary conditions so that they disappear asymptotically in forward time.

In [4] Lax and Phillips consider the acoustic equation in an exterior domain $G \subset \mathbb{R}^n$:

\[
\begin{aligned}
  u_{tt} &= \Delta u & \text{in } G, \\
  \partial_n u + \sigma u_t &= 0, & \sigma > 0 \text{ in } \partial G.
\end{aligned}
\]

They assume $G$ contains the complement of the ball of radius $\rho$. As in [4], we define $H$ to be the Hilbert space of all initial data $d$ with finite energy in $G$. Let $T(t)$ be the (strongly continuous) contraction semigroup formed by mapping initial data into data at time $t$.

If $G = \mathbb{R}^n$ (and the second part of (1.1) is vacuous) we will denote $H$ by $H_0$ and $T(t)$ by $U_0(t)$. We note that $U_0(t)$ is a unitary group. We denote the cogenerator (see Chapter 3 of [5]) of $T(t)$ by $T$ and the cogenerator of $U_0(t)$ by $U_0$. Let $D_+ \subset H$ be the set of all initial data vanishing on $\{x| |x| < \rho \pm t, t > 0\}$.

We will prove the following

**Theorem.** Let $n$ be greater than 1. (Recall that $G \subset \mathbb{R}^n$.) If $d \in D_-$ and $d \not\equiv 0$. Then $\lim_{t \to +\infty} T(t)d \neq 0$.

Before starting the proof we recall some of the material in [2], [3], and [4]. We represent the action of $U_0(t)$ on $H_0$ as right translation on $L^2(\mathbb{R}, N)$ (i.e., the set of all square integrable $N$-valued functions on $\mathbb{R}$) for some auxiliary Hilbert space $N$ so that $D_-$ is mapped onto $L^2(\mathbb{R}_- - \rho, N)$. In this representation $D_+$ is mapped onto

\[ L^2(\mathbb{R}_+ + \rho, N) \text{ if } n \text{ is odd} \]

and

\[ \mathcal{K}L^2(\mathbb{R}_+ + \rho, N) \text{ if } n \text{ is even} \]

where

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(1.2) \[ \mathcal{K}(s) = \mathcal{T}^{-1} \mathcal{K}(\sigma) \mathcal{T}, \]
(1.3) \[ \mathcal{K}(\sigma) = \text{sgn} \sigma \]
and \( \mathcal{T} \) is the Fourier transform.

Since \( T(t)|_{D_+} = U_0(t)|_{D_+} \) for \( t > 0 \), and \( T(t)|_{D_-} = U_0(-t)|_{D_-} \) for \( t > 0 \), we can embed \( H \) onto \( L^2(\mathbb{R}, N) \) so that \( T(t)^* \) acts on \( L^2(\mathbb{R}_- - \rho, N) \) as left translation by \( t \) and \( T(t) \) acts on \( L^2(\mathbb{R}_+ + \rho, N) \) (resp. \( \mathcal{K}L^2(\mathbb{R}_+ + \rho, N) \)) if \( n = \text{odd} \) (resp. if \( n = \text{even} \)) as right translation by \( t \). The action of \( T(t) \) on the rest of \( L^2(\mathbb{R}, N) \) is more difficult to describe.

**Lemma 1.1.** Let \( f(s) \in D_- \). Then \( f(s) \in T^*D_- \) if and only if \( \hat{f}(\sigma) \), the Fourier transform of \( f(s) \), is zero at the point \((0, -i)\).

**Proof.** Let \( f(s) \in D_- \). Then by Chapter III of [5] and the fact that \( T(t)^*f(s) = f(s + t) \) for \( t \in \mathbb{R}_+ \) we conclude

\[
(T^*f)(s) = f(s) \operatorname{s-lim}_{t \to 0^+} \frac{t}{1 + t} \sum_{n=0}^{\infty} \frac{f(s + nt)}{(1 + it)^n}.
\]

Taking the Fourier transform

\[
\mathcal{F}(T^*f)(\sigma) = \mathcal{F}(\hat{f})(\sigma) + \operatorname{s-lim}_{t \to 0^+} \frac{t}{1 + t} \sum_{n=0}^{\infty} \frac{e^{int}\hat{f}(\sigma)}{(1 + it)^n} = \mathcal{F}(\hat{f})(\sigma)(1 - 1/\sigma).
\]

Since \( \mathcal{F}(T^*f)(\sigma) \) and \( \mathcal{F}(\hat{f})(\sigma) \) are analytic in the lower half plane, the above calculation shows \( (Tf)(-i) = 0 \).

Conversely if \( g(s) \in D_- \) and \( \hat{g}(\sigma) \) has a zero at \(-i\), then

\[
\hat{g}(\sigma) = (\sigma + i)(\sigma - i)^{-1} \hat{f}(\sigma) \quad \text{for some } f \in D_-.
\]

But \( T^* = U_0^{-1} \) on \( D_- \), and \( U_0^{-1} \) acts as multiplication by \((\sigma + i)/(\sigma - i)^{-1}\) in the Fourier transform of the translation representation (called the spectral representation in [2]). To see this, note that \( A_0 \), the generator of \( U_0(t) \), acts as multiplication by \( it \) in the spectral representation. The action of

\[
U_0 = (I + A_0)(I - A_0)^{-1}
\]

is now clear. Thus \( g(s) = (T^*f)(s) \) for \( f \in D_- \). This proves the lemma.

Define the wave operators as

\[
(1.4) \quad W_1 = \operatorname{s-lim}_{t \to \infty} T(t)J_0U_0(-t), \quad W_2 = \operatorname{s-lim}_{t \to \infty} U_0(-t)JT(t),
\]

where \( J, J_0 \) are continuous linear maps from \( H \) to \( H_0 \) and \( H_0 \) to \( H \) respectively which act as the identity on \( D_- \vee D_+ \). Define the scattering operator \( S \) as in [4] by

\[
(1.5) \quad S = W_2W_1.
\]

**Lemma 1.2.** For any \( d \in D_- \)

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(1.6) \[ P_{D^*} Td = P_{D^*} U_0 Sd. \]

PROOF. From the definitions of \( W_1 \) and \( W_2 \) we have \( W_2 T = U_0 W_1 \). Since \( W_2 |_{D^*} = I|_{D^*} = W_2^* |_{D^*} \), we see that for any \( d \in H \),
\[ P_{D^*} U_0 W_2 d = P_{D^*} W_2 T d = W_2 P_{D^*} T d = P_{D^*} T d. \]
If \( d \in D_- \) we see that \( W_1 d = d \) so that by (1.5)
\[ P_{D^*} T d = P_{D^*} U_0 [W_2 W_1] d = P_{D^*} U_0 Sd \]
for any \( d \in D_- \). Q.E.D.

Since \( U_0(t) \) acts as right translation by \( t \) on \( L^2(\mathbb{R}, \mathbb{N}) \) we can calculate \( U_0 \) as

(1.7) \[ (U_0 f)(s) = f(s) - 2e^{-\xi} \int_{-\infty}^s f(\xi) e^{i\xi} d\xi, \quad f \in L^2(\mathbb{R}, \mathbb{N}). \]

The operator \( S \) on \( H_0 = L^2(\mathbb{R}, \mathbb{N}) \) commutes with \( U_0(t) \) (= translation by \( t \)) and it follows that in the spectral representation (= Fourier transform space) the corresponding operator, denoted by \( \hat{S} \), acts on \( L^2(\mathbb{R}, \mathbb{N}) \) by multiplication

\[ \hat{S} f(\sigma) = \hat{S}(\sigma) f(\sigma), \quad f \in L^2(\mathbb{R}, \mathbb{N}). \]

We now prove the theorem for the case when \( n \) is odd \((\neq 1)\). In [4] it is shown that \( \hat{S}(\sigma) \) has an analytic extension to the lower half plane if \( n \) is odd. In particular it is shown that

(1.8) \[ S(L^2(\mathbb{R} - \rho, \mathbb{N})) \subset L^2(\mathbb{R} + \rho, \mathbb{N}). \]

PROPOSITION 1.3. Let \( d \) be a nonzero element of \( D_- \). We also assume \( U_0 d \notin D_- \) and \( \hat{S}(-i) \) is invertible.
Then \( U_0 Sd \) is not orthogonal to \( D_+ \).

PROOF. Let \( d \in D_- \). Then in the translation representation \( d \) has its support in \((-\infty, -\rho]\). Since \( S \) satisfies (1.8) we see \( (Sf) \) has its support in \((-\infty, \rho]\). From (1.7) it is clear that if \( U_0 Dd \) has its support in \((-\infty, \rho]\) then

(1.10) \[ 0 = \int_{-\rho}^0 (Sd)(\xi) e^{i\xi} d\xi = \int_{-\infty}^{\infty} (Sd)(\xi) e^{i\xi} d\xi. \]

Rewriting (1.10) we see \( \hat{S} \hat{d} = 0 \), i.e. \( \hat{S}(-i) \hat{d} = 0 \). By assumption, \( \hat{S}(-i) \) is invertible and we conclude \( \hat{d}(-i) = 0 \). Thus by Lemma 1.1 we see \( d \in T^* D_- = U_0^{-1} D_- \), i.e. \( U_0 d \in D_- \). But we assumed \( U_0 d \notin D_- \). Thus \( U_0 Sd \) does not have its support in \((-\infty, \rho]\).

Since \( D_+ = L^2(\rho, \infty, \mathbb{N}) \) in the translation representation, we conclude that \( U_0 Sd \) is not orthogonal to \( D_+ \).

PROPOSITION 1.4. Let \( d \in D_- \) be nonzero and assume (1.9) holds. Then \( \text{s-lim}_{t \to \infty} T(t)d \neq 0 \).

PROOF. By Proposition III 9.1 of [5], it suffices to show
Now if $d \not\equiv 0$ we can find a smallest $m > 0$ so that $T^m d \not\in T^* D_-$ and $T^m d \in D_-$. We conclude by Proposition 1.3 that $U_0 S T^m d$ is not orthogonal to $D_+$. Thus by (1.6) we see $P_{D_+} T^{m+1} d \neq 0$. Now let $U$ on $K \supset H$ be the minimal unitary dilation of $T$ (see [5]). On $D_+$ we see $T|_{D_+} = U|_{D_+} = U|_{D_+}$. Thus for $n > 0$

$$0 = (D_+, H \otimes D_+) = (U^n D_+, (H \otimes D_+))
= (T^n D_+, U^n (H \otimes D_+)) = (T^n D_+, T^n (H \otimes D_+)).$$

Thus if $T^{m+1} d = \beta \oplus \beta_+$, $\beta \in H \otimes D_+$, $\beta_+ \in D_+$ we see

$$T^\ast \beta_+ \perp T^\ast \beta \quad \text{all } n > 0.$$

Thus

$$\|T^\ast T^{m+1} d\|^2 = \|T^\ast \beta_+ + T^\ast \beta\|^2 = \|T^\ast \beta_+\|^2 + \|T^\ast \beta\|^2 > \|T^\ast \beta_+\|^2 = 0.$$

Thus we can conclude (1.11). Q.E.D.

We now relax the restriction imposed by (1.9) and complete the proof of the theorem in the odd-dimensional case.

**Proposition 1.5.** If $d \in D_-$, then

$$\lim_{t \to \infty} T(t) d \neq 0.$$  

**Proof.** Recall that $G$ contains the complement of the ball of radius $\rho$. Define $V(x, t) = u(cx, ct)$, $c > 0$. Then $u(x, t)$ satisfies

$$\begin{cases}
\frac{\partial v}{\partial t} = \Delta v & \text{in } G', \\
\frac{\partial v}{\partial t} + \sigma v = 0 & \text{in } \partial G', \sigma > 0,
\end{cases}$$

where $G' = \{c^{-1} [g] | g \in G \}.$

Define $D_-(v)$ as the subspace of initial data which vanishes on $\{|x| \leq \rho/c + t, t < 0\}$ under the action of (1.13). Recall the definition of $D_-(u)$ as the subspace of initial data which vanishes on $\{|x| < \rho + t, t < 0\}$ under the action of (1.1). It is clear that $c$ can be chosen so that $G'$ contains the complement of a ball of radius less than one. By Theorem 10.10 of [4], since $n$ is greater than one, we can conclude that the scattering matrix for the $v$-system is invertible at $-i$. Thus by Propositions 1.3 and 1.4, (1.12) holds for the $v$-system. But the statement of the theorem is invariant under the change from the $v$ to the $u$ systems. Thus (1.12) holds for both the $u$ and $v$ systems and the theorem is proven for the case when $n$ is odd and greater than one.

We now look at the case when $n$ is even. To prove the theorem in this case it suffices to establish that $U_0 S d$ is not orthogonal to $D_+$ for any nonzero $d$ in $D_-$. Once this is done the argument in Proposition 1.4 (with $m = 0$) can be used as before to conclude (1.11).
Proposition 1.6. Let \( d \) be a nonzero element of \( D_- \) and let \( n \) be even. Then \( U_0Sd \) is not orthogonal to \( D_+ \).

Proof. Let \( d \in D_- \). Then 
\[
(Sd, D_+) = (W_2W_1d, D_+) = (W_1d, W_2^*D_+) = (d, D_+).
\]
If \( Sd \) is orthogonal to \( D_+ \), then \( d \in D_- \cap D_+ \perp \). Thus \( \hat{d}(\sigma) \) and \( \hat{\kappa}(\sigma) \cdot \hat{d}(\sigma) \) both have analytic extensions to the lower half plane. But this is clearly impossible unless \( \hat{d}(\sigma) \equiv 0 \), i.e. \( d(s) \equiv 0 \). Thus \( Sd \) is not orthogonal to \( D_+ \). Since \( U_0^{-1}D_+ \supset D_+ \) we conclude \( U_0Sd \) is not orthogonal to \( D_+ \).

The proof of the theorem is now complete.

In conclusion, I would like to thank the referee for pointing out that the theorem does not hold for \( n = 1 \), by providing the following counterexample:

\[
G = \{ x > a \}, \quad u = f(x + t), \quad f \text{ of compact support},
\]
\[
- u_x + u_t = 0 \quad \text{on} \quad x = a.
\]

References


