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ADDENDUM TO "ARITHMETIC MEANS OF FOURIER COEFFICIENTS"

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ABSTRACT. Let f be integrable and periodic with period 2π. Then a necessary and sufficient condition for $\tilde{f}$ to be equivalent to a continuous function is that $-(1/\pi)\int_t^\pi (f(x+u)-f(x-u))/2\tan(u/2)\,du$ converges uniformly in x as $t \to 0+$.

In what follows, we shall not distinguish between equivalent functions.

Zamansky [2] (also see [3, Volume 1, p. 180, Exercise 5(a)]) proved the following.

Let f be continuous and periodic. Then a necessary and sufficient condition for $\tilde{f}$ to be continuous is that

$$\tilde{f}(x;t) = -\frac{1}{\pi}\int_t^\pi \frac{f(x+u)-f(x-u)}{2\tan u/2}\,du \tag{1}$$

converges uniformly as $t \to 0+$.

In this note we show that the restriction of continuity on f can be dropped in the above theorem. Equivalently we show the following.

THEOREM. *Let f be integrable and periodic. Then a necessary and sufficient condition for $\tilde{f}$ to be continuous is that $\tilde{f}(x;t)$ converges uniformly in x as $t \to 0+$, where $\tilde{f}(x;t)$ is defined in* (1).

PROOF. Sufficiency is obvious since $\lim_{t\to 0+}\tilde{f}(x;t) = \tilde{f}(x)$ for a.e. x.

Conversely, let $\tilde{f}$ be continuous. Take $\tilde{f} = g$ and $\tilde{f}(x;t) \equiv g(x;t)$. Let T_S and T_H be defined as in [1]. Also let $g_x(t) \equiv g(t+x)$. Now define $T_x g = T_S(g_x)$. Then T_x is a bounded operator on L^∞ by [1, Lemma 1.1].

Also using [1, (1.1)] with $f_c(t)$ replaced by $h_x(t) \equiv [g_x(t) + g_x(-t)]/2$, we can show that

$$T_x g(t) = T_S h_x(t) = T_H \tilde{h}_x(t) = \tfrac{1}{2}[\pi g(x;t) + G_x(t)] \tag{2}$$

for a.e. $t > 0$ where $G_x(t) \equiv \sum_{n=1}^\infty [A_n(x;g)/n]\sin nt$.

It can easily be shown that the family $\{T_x g \mid x \in [0,2\pi]\}$ is a normal family since $[0,2\pi]$ is compact and

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$$x_n \to x \Rightarrow \|g_{x_n} - g_x\|_\infty \to 0 \Rightarrow \|T_S g_{x_n} - T_S g_x\|_\infty \to 0$$
$$\Rightarrow \|T_{x_n} g - T_x g\|_\infty \to 0.$$

Hence, by the Ascoli-Arzela Theorem, the family $\{T_x g \mid x \in [0, 2\pi]\}$ is equicontinuous. Therefore $\{T_x g(t)\}$ converges uniformly in x as $t \to 0+$.

Similarly we can show that the family $\{G_x(t) \mid x \in [0, 2\pi]\}$ is also equicontinuous and, hence, $G_x(t)$ converges uniformly in x as $t \to 0+$. Therefore, by (2), $g(x; t)$ converges uniformly in x as $t \to 0+$. Hence the result.

References

1. R. Sinha, *Arithmetic means of Fourier coefficients*, Proc. Amer. Math. Soc. **55** (1976), 83–86.

2. M. Zamansky, *Sur l'approximation des fonctions continues périodiques*, C. R. Acad. Sci. Paris **228** (1949), 460–461. MR **10**, 449.

3. A. Zygmund, *Trigonometric series*, 2nd ed. reprinted, Vols. I, II, Cambridge Univ. Press, New York, 1968. MR **38** #4882.

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