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A FACTORABLE BANACH ALGEBRA WITH INEQUIVALENT REGULAR REPRESENTATION NORM

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ABSTRACT. An example is given of a semisimple commutative Banach algebra which factorizes but whose norm is not equivalent to the norm induced by its regular representation. This is a stronger version of the example given in [4] and it can be viewed as an example of a factorizing commutative abstract Segal algebra.

Let S be the semigroup (with pointwise addition) of all real sequences $b = \{b_n\}_{n\in\mathbf{N}}$ with $b_n > 0$ for almost all n. For $b = \{b_n\}_{n\in\mathbf{N}} \in S$ define $w(b) = \min\{n \in \mathbf{N} | b_m > 0 \text{ for all } m \geqslant n\}$. Since $w(b + c) = n$ implies $b_{n-1} \leqslant 0$ or $c_{n-1} \leqslant 0$, we have $w(b + c) \leqslant w(b)w(c)$ for all $b, c \in S$. Let $A_w = l^1(S, w)$ be the weighted semigroup algebra with norm $\| \ \|_w$:

$$\left\| \sum_{n\in\mathbf{N}} \lambda_n \varepsilon_{f_n} \right\|_w = \sum_{n\in\mathbf{N}} |\lambda_n| \cdot w(f_n),$$

where ε_{f_n} denotes the Dirac measure concentrated at $f_n \in S$.

Let $a = \sum_n \lambda_n \varepsilon_{f_n} \in A_w$, with $f_n = \{f_{nk}\}_{k\in\mathbf{N}}$, and let $\varepsilon > 0$ be given. For each $m \in \mathbf{N}$ let r_m be such that $\|\sum_{n>r_m} \lambda_n \varepsilon_{f_n}\|_w < \varepsilon/(m+1)^3$. We may suppose $r_m < r_{m+1}$ for all m. Let $K_m = \{f_{nm} | 1 \leqslant n \leqslant r_m, f_{nm} > 0\}$ and define

$$k_m = \begin{cases} \frac{1}{2} \min K_m, & K_m \neq \varnothing, \\ 1, & K_m = \varnothing. \end{cases}$$

Let $k = \{k_m\}_{m\in\mathbf{N}}$ and $g_n = f_n - k$ for all n. For $m, n \in \mathbf{N}$ with $n \leqslant r_m$ we have $g_{nm} > 0$ if and only if $f_{nm} > 0$. This implies $g_n \in S$ and $w(g_n) \leqslant m \cdot w(f_n)$. Hence

$$\begin{aligned} \left\| \sum_n \lambda_n \varepsilon_{g_n} \right\|_w &\leqslant \left\| \sum_{n=1}^{r_1} \lambda_n \varepsilon_{g_n} \right\|_w + \sum_{m=1}^{\infty} \left\| \sum_{n=r_m+1}^{r_{m+1}} \lambda_n \varepsilon_{g_n} \right\|_w \\ &\leqslant \|a\|_w + \sum_m \sum_{r_m<n} |\lambda_n| \cdot (m+1) w(f_n) \\ &\leqslant \|a\|_w + \varepsilon \cdot \sum_m \frac{1}{(m+1)^2}. \end{aligned}$$

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So a can be factored in A_w: $a = \varepsilon_k \cdot (\sum_n \lambda_n \varepsilon_{g_n})$ with $\|\sum_n \lambda_n \varepsilon_{g_n}\|_w \leqslant \|a\|_w + \varepsilon'$ which means that factorization can be achieved almost multiplicatively for the norms. It is easily seen that if we had, in addition, the possibility of choosing the factor $\sum_n \lambda_n \varepsilon_{g_n}$ arbitrarily close to a, then A_w would in fact have bounded approximate units. However, the situation is quite different:

For $n \in \mathbf{N}$ define $d_n = \{d_{nk}\}_{k \in \mathbf{N}} \in S$ by

$$d_{nk} = \begin{cases} 0, & k < n, \\ 1, & k \geqslant n. \end{cases}$$

Obviously, $w(d_n) = n$. Since $d_{nk} \geqslant 0$ for all $k \in \mathbf{N}$, the operator of left multiplication by ε_{d_n} in A_w has norm $\leqslant 1$. This proves that the norm $\| \; \|_w$ on A_w is not equivalent to the norm induced by the regular representation.

COROLLARY. *There is a nontrivial commutative abstract Segal algebra* (*see* [3]) *which factorizes.*

PROOF. Let $A = A_w$ be as above and $B(A)$ the Banach algebra of all bounded linear operators on A. Let B be the norm closure of A in $B(A)$ where, of course, A is embedded in $B(A)$ by its regular representation. Clearly, B is a Banach algebra and A is a dense ideal in B since all elements of B are multipliers of A. The required norm inequalities are obviously satisfied and, as follows from above, $B \neq A$.

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