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AN ELEMENTARY PROOF OF A FINITE RIGIDITY PROBLEM BY INFINITESIMAL RIGIDITY METHODS[1]

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ABSTRACT. Let two compact, isometric surfaces with boundary be given having positive gauss curvature. If the surfaces can be placed so that their normal spherical images lie in a compact subset of a hemisphere of the unit sphere and so that the isometry is the identity on the boundary then the isometry is the identity mapping.

The proof is elementary in the sense that no integral formulae or maximum principles for elliptic operators are needed.

An example is given of a surface satisfying the above hypotheses which is neither convex nor has a representation in the form $z = f(x, y)$.

1. **Introduction**. The purpose of this paper is to provide an elementary proof by methods of infinitesimal rigidity of the following result:

Let two compact, isometric surfaces with boundary be given having positive gauss curvature. If the surfaces can be placed so that their normal spherical images lie in the same hemisphere of the unit sphere and so that the isometry is the identity on the boundary, then the isometry is the identity mapping.

The proof is elementary in the sense that no integral formulae or maximum principles for elliptic operators are needed. The facts quoted from [2] in §3 are derived from straightforward calculations such as occur in a basic text in classical differential geometry of surfaces, for example [4]. However, it should be noted that surfaces satisfying the above hypotheses need not be convex nor have a representation in the form $z = f(x, y)$ so that certain standard methods could not be used, at least in their simplest form (see §5).

Infinitesimal methods have been used by Cohn-Vossen [1] to prove the unique determination of an ovaloid by its metric and by Pogorelov [5] to prove the general monotypy theorem for convex surfaces.

The infinitesimal methods enter as follows: If X' and X'' are position vectors of two isometric surfaces then the condition $dX'^2 = dX''^2$ is equivalent to $dX \cdot dZ = 0$ where $X = \frac{1}{2}(X'' + X')$ and $Z = \frac{1}{2}(X'' - X')$. If Z is

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interpreted as a deformation field on the "mean surface" X then $dX \cdot dZ = 0$ is just the condition that Z be an infinitesimal bending field on the surface X. The methods developed in [3] are then used to show that Z is trivial.

It would be nice to prove the uniqueness theorem for ovaloids by the present methods. The difficulty is that the mean surface need not be regular for certain positions of the surfaces in space.

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2. **Precise formulation of the problem.** All manifolds will be of class C''. A manifold with boundary will be assumed to be immersed in a manifold. Likewise when we speak of a manifold with compact closure $\overline{M}$, it will be understood that $\overline{M}$ is immersed in a manifold. Any map f on such objects is to be understood as a restriction from the containing manifold.

Let M be a compact two-dimensional, orientable Riemannian manifold with boundary, having gauss curvature $K > 0$.

Let f': $M \to E^3$ and f'': $M \to E^3$ be isometric immersions (also called surfaces), i.e., locally one-one mappings of M in E^3 of rank two such that the metric induced on the images by E^3 equals the metric of M.

Represent the surfaces by position vectors $X' = f'(x)$ and $X'' = f''(x)$ for $x \in M$. Then $dX'^2 = dX''^2$ at x.

To fix our ideas we suppose the unit surface normals n' and n'' are taken to point toward the concave sides of the surfaces.

MAIN RESULT. *If*

1. $f'(x) = f''(x)$ *for* $x \in \partial M$,
2. *there exists a constant vector* e *such that* $e \cdot n' > 0$ *and* $e \cdot n'' > 0$,

then $f'(x) \equiv f''(x)$ *for* $x \in M$.

3. **General considerations of infinitesimal bending theory.** We here follow Efimov [2, pp. 53–57]. Let Z be a C'' vector field defined on a surface S in E^3.

DEFINITION. Z is a *bending field* on S if $dZ \cdot dX = 0$.

If u, v are local parameters on S this is equivalent to the differential equations

$$X_u \cdot Z_u = 0, \quad X_u \cdot Z_v + X_v \cdot Z_u = 0, \quad X_v \cdot Z_v = 0, \tag{3.0}$$

where $X_u = \partial X / \partial u$, etc. This is equivalent to the fact that the first fundamental form of the family of surfaces $X(u, v, t)$ is constant to first order in t at $t = 0$, under suitable differentiability hypotheses (with $Z = \partial X / \partial t$ at $t = 0$).

It is known that if the surface is regular then a unique C' vector field Y (called the *rotation field*) exists such that $Y \times dX = dZ$, or equivalently,

$$Y \times X_u = Z_u, \qquad Y \times X_v = Z_v. \tag{3.1}$$

Further, there exist scalars α, β, γ such that

(3.2) $$Y_u = \alpha X_u - \beta X_v, \qquad Y_v = \gamma X_u - \alpha X_v.$$

If Z is taken as a position vector it describes a surface called the *bending surface*. Likewise Y describes the *rotation surface*. Both of these surfaces may be singular even when S is everywhere regular, which is the usual assumption in infinitesimal bending theory. In the present applications this assumption is not made for S. At singular points of S, Y is not defined.

A bending field Z is said to be *trivial* if it is of the form $Z = C \times X + D$ where C, D are constant vectors. Thus a trivial bending field coincides with the velocity field of a rigid motion of S.

Two bending fields which differ by a trivial bending field are called *equivalent*.

If a surface admits only trivial bendings it is called *infinitesimally rigid*.

4. **Proof of Main Result.**

LEMMA 1. *Let M be a two-dimensional orientable manifold with compact closure $\overline{M}$. Let f: $\overline{M} \to E^3$ be an immersion of $\overline{M}$ such that*:

1. *$f(\overline{M})$ is a surface having gauss curvature $K > 0$ in the metric induced by E^3;*

2. *There exists a constant vector e such that $e \cdot n > 0$, where n is the unit surface normal of $f(\overline{M})$ (hemisphere condition);*

3. *A bending field Z exists on $f(\overline{M})$ with corresponding rotation field Y.*

Then $Y \cdot n \neq 0$ on the frontier of M implies $Y \cdot n \neq 0$ on M.

PROOF. First we prove the lemma under the assumption that $Y \neq 0$ on M. Suppose, by way of contradiction, that there existed a point x of M at which $Y \cdot n = 0$. Not both $(Y \cdot n)_u = 0$ and $(Y \cdot n)_v = 0$ at x, for, by (3.2), it follows that $Y \cdot n_u = 0$ and $Y \cdot n_v = 0$. Since $K > 0$, n_u, n_v and n are linearly independent, so Y would vanish.

Hence, the implicit function theorem implies that an open, smooth C'' curve segment exists on M, containing x, on which $Y \cdot n = 0$. The endpoints of the segment also lie in M by hypothesis and the argument can be applied again to extend the segment indefinitely in both directions to a simple curve σ which stays away from the frontier of M (i.e., is contained in a compact subset of M).

$f(\sigma)$ may be represented by a vector $X = X(s)$, $-\infty < s < +\infty$, where s is arclength.

Since $K > 0$, $(dX/ds) \cdot (dn/ds) \neq 0$. This, together with $Y \cdot dn/ds = 0$, implies that $Y \times dX/ds \neq 0$ on σ.

Let λ be defined on σ by $Y \times dX/ds = \lambda n$. λ never vanishes and simple compactness arguments show that λ and $e \cdot n$ are bounded away from zero on σ. Hence

$$\frac{d(e \cdot Z)}{ds} = \frac{dZ}{ds} \cdot e = Y \times \frac{dX}{ds} \cdot e = \lambda e \cdot n \gg 0.$$

Thus $e \cdot Z$ would be unbounded on the compact set $\overline{M}$. Thus the assumption that $Y \cdot n = 0$ leads to a contradiction.

We conclude the proof of Lemma 1 by showing that if Y is *any* rotation vector satisfying the hypothesis, then a rotation vector exists which satisfies the same hypothesis but never vanishes.

The spherical image mapping, $n: f(\overline{M}) \to U =$ the unit sphere, takes $\overline{M}$ to the compact subset of the open hemisphere of U defined by hypothesis 2, i.e., $\{X \in U: e \cdot X > 0\}$. Hence if g is any nonzero vector sufficiently close in direction to e then $(n \circ f)\overline{M}$ lies in the hemisphere $\{X \in U: g \cdot X > 0\}$. In other words there exists an open solid half-cone C of vectors g such that $g \cdot n > 0$ where $n = (n \circ f)x$ for $x \in \overline{M}$, $g \in C$.

By hypothesis, $Y \cdot n \neq 0$ on the frontier of M. Suppose for example $Y \cdot n > 0$ there. Since the frontier of M is compact, $Y \cdot n$ has a positive minimum, m, there.

The rotation surface R described by the position vector Y cannot contain an open subset of E^3. Hence there exists a constant vector $g_0 \in C$ such that $Y - g_0 \neq 0$ for any $Y \in R$ and such that $|g_0| < m$. Hence we have $Y \cdot n \geqslant m > g_0 \cdot n > 0$ on the frontier of M.

It follows that $Y - g_0$ is a rotation vector for the bending field $Z - g_0 \times X$ (equivalent to Z) such that $(Y - g_0) \cdot n > 0$ on the frontier of M. By the first part of our proof $(Y - g_0) \cdot n > 0$ on $\overline{M}$. Hence $Y \cdot n > 0$ on $\overline{M}$.

Similarly the assumption $Y \cdot n < 0$ on the frontier of M leads to $Y \cdot n < 0$ on $\overline{M}$.

NOTE. The proof yields a somewhat stronger conclusion: If $Y \cdot n \gtrless 0$ on the frontier of M then there exists a vector g_0 such that $Y \cdot n \gtrless g_0 \cdot n \gtrless 0$ on M.

LEMMA 2. *Let the hypothesis be as in Lemma* 1 *except that f need not be of rank* 2 *on the frontier of M. Then $Z_u \times Z_v \neq 0$ on the frontier of M implies $Z_u \times Z_v \neq 0$ on M.*

PROOF. At any point of M (but not necessarily on the frontier of M, since Y need not be defined there)

$$Z_u \times Z_v = (Y \times X_u) \times (Y \times X_v) = (Y \cdot X_u \times X_v)Y.$$

Thus $Z_u \times Z_v = 0$ on M if and only if $Y \cdot n = 0$.

By continuity of $Z_u \times Z_v$ there is a closed neighborhood N of the frontier of M such that $Z_u \times Z_v \neq 0$ there and such that the hypotheses of Lemma 1 are satisfied by $Y \cdot n$ on $\overline{M} - \overline{M} \cap N$.

PROOF OF MAIN RESULT. Define $X = \frac{1}{2}(X' + X'')$, $Z = \frac{1}{2}(X'' - X')$. As mentioned in the introduction, $dX \cdot dZ = 0$, so Z is a bending field on the surface X, which may be singular.

$$\begin{aligned} X_u \times X_v &= \tfrac{1}{2}(X'_u + X''_u) \times \tfrac{1}{2}(X'_v + X''_v) \\ &= \tfrac{1}{4}(X'_u \times X'_v + X''_u \times X'_v + X'_u \times X''_v + X''_u \times X''_v), \end{aligned}$$

and

$$Z_u \times Z_v = \tfrac{1}{2}(X_u'' - X_u') \times \tfrac{1}{2}(X_v'' - X_v')$$
$$= \tfrac{1}{4}(X_u' \times X_v' - X_u'' \times X_v' - X_u' \times X_v'' + X_u'' \times X_v''),$$

so

$$X_u \times X_v + Z_u \times Z_v = \tfrac{1}{2}(X_u' \times X_v' + X_u'' \times X_v''),$$

hence

$$e \cdot X_u \times X_v + e \cdot Z_u \times Z_v > 0 \quad \text{on } M \tag{4.0}$$

follows from hypothesis 2. But $e \cdot X_u \times X_v > 0$ on ∂M since $Z_u \times Z_v = 0$ there. Hence there is an open set U of M such that $e \cdot X_u \times X_v > 0$ on U and U is *adjacent* to ∂M, that is, part of the frontier of U is ∂M. Thus X is regular on U.

Let U^* be the maximal open subset of M on which $e \cdot X_u \times X_v > 0$. The frontier of U^* is the union of two disjoint sets, ∂M and a set A on which $e \cdot X_u \times X_v = 0$. A may be empty, in which case $U^* = M$.

Define $Z_\lambda = Z + \lambda e \times X$ where λ is an arbitrary real number. Z_λ is a bending field on X equivalent to Z.

$$\begin{aligned}(Z_\lambda)_u \times (Z_\lambda)_v &= (Z_u + \lambda e \times X_u) \times (Z_v + \lambda e \times X_v)\\ &= Z_u \times Z_v\\ &\quad + \lambda[(Z_u \cdot X_v)e - (Z_u \cdot e)X_v + (Z_v \cdot e)X_u - (Z_v \cdot X_u)e]\\ &\quad + \lambda^2(e \cdot X_u \times X_v)e.\end{aligned} \tag{4.1}$$

The linear dependence of Z_u and Z_v on ∂M together with (3.0) imply that on ∂M,

$$(Z_\lambda)_u \times (Z_\lambda)_v = \lambda[(Z_v \cdot e)X_u - (Z_u \cdot e)X_v] + \lambda^2(e \cdot X_u \times X_v)e.$$

Now X_u, X_v, e are linearly independent on ∂M so

$$(Z_\lambda)_u \times (Z_\lambda)_v \neq 0 \quad \text{on } \partial M \text{ if } \lambda \neq 0. \tag{4.2}$$

From (4.0) it follows that on A, $Z_u \times Z_v \neq 0$. Since A is compact $|Z_u \times Z_v| \gg 0$ on A. Since $(Z_\lambda)_u \times (Z_\lambda)_v$ is a uniformly continuous function of λ on A it follows that $(Z_\lambda)_u \times (Z_\lambda)_v \neq 0$ on A for $|\lambda|$ sufficiently small. Hence, by Lemma 2,

$$(Z_\lambda)_u \times (Z_\lambda)_v \neq 0 \quad \text{on } U^* \text{ for small } |\lambda| \text{ if } \lambda \neq 0. \tag{4.3}$$

$U^* \cup \partial M$ is a regular surface with boundary, represented by the position vector X, so a unique rotation vector Y corresponding to the bending field Z exists such that $Y \times dX = dZ$. Since

$$dZ_\lambda = dZ + \lambda e \times dX = (Y + \lambda e) \times dX,$$

$Y + \lambda e$ is the unique rotation vector for Z_λ.

$$\begin{aligned}(Z_\lambda)_u \times (Z_\lambda)_v &= [(Y + \lambda e) \times X_u] \times [(Y + \lambda e) \times X_v]\\ &= [(Y + \lambda e) \cdot (X_u \times X_v)](Y + \lambda e).\end{aligned} \tag{4.4}$$

By (4.3) the last term in square brackets, call it q_λ, does not vanish on U^* if $|\lambda|$ is small but not zero.

But $Y \cdot X_u \times X_v = 0$ on ∂M (put $\lambda = 0$ in (4.1) and in (4.4)) so q_λ is positive or negative on ∂M, and hence near ∂M, according as λ is positive or negative.

Since q_λ does not vanish on U^* it also is positive or negative on all of U^* according as λ is. Hence, by continuity, $q_0 \equiv 0$ on U^*.

This is equivalent to $Y \cdot n \equiv 0$, where $n = X_u \times X_v / |X_u \times X_v|$. It follows from (3.2) that $Y \cdot n_u \equiv 0$ and $Y \cdot n_v \equiv 0$.

Now it is known (cf. introduction) that the gauss curvature of the mean surface X is positive at regular points if the gauss curvatures of X' and X'' are. Therefore n, n_u, n_v are linearly independent, hence $Y \equiv 0$. By (3.1) $Z_u \equiv 0$ and $Z_v \equiv 0$ so Z is constant on U^*. Since $Z = 0$ on ∂M, $Z \equiv 0$ on U^*. By (4.0) A is empty, hence $U^* = M$ and $Z \equiv 0$ on M.

5. **An example of a nonconvex surface satisfying the hypotheses**. There is a tubular surface constructed on the cylindrical helix which satisfies the conditions of §2 but is neither convex nor has a representation of the form

$$z = f(x, y).$$

Let

$$X(t) = a(i \cos t + j \sin t + ctk)$$

where i, j, k are orthonormal basis vectors and a and c are positive constants. This is a vector representation of a helix lying on a cylinder of radius a. Let $X(v)$ be a representation of the helix in terms of arclength v and let v_1, v_2, v_3 be the Frenet triple of the helix, and r a positive constant. Then the surface

$$X(u, v) = X(v) + r(v_2(v) \cos u + v_3(v) \sin u)$$

is a tubular surface (or canal surface) whose cross-sections by the normal planes of the helix are circles of radius r (cf. [4, p. 76, Exercise 6.7.4]).

Elementary calculation using the Frenet equations yields

$$X_u \times X_v = r(1 - r\kappa \cos u)(v_3(v) \sin u + v_2(v) \cos u)$$

so the surface is regular if $r < 1/\kappa$, where κ is the curvature of the helix. Also,

$$k \cdot X_u \times X_v = r(1 - r\kappa \cos u)\left(a(a^2 + c^2)^{-1/2} \sin u\right)$$

and the gauss curvature is $K = -\kappa \cos u/\sqrt{g}$. Thus if we restrict u to satisfy $\pi + \varepsilon \leqslant u \leqslant \frac{3}{2}\pi - \varepsilon$ where ε is a small positive number we obtain a surface whose closure has positive gauss curvature and has its spherical image in a hemisphere. If v is restricted to a closed interval then all of the conditions of §2 are satisfied. Such a surface is clearly not convex and cannot be represented in the form $z = f(x, y)$ if the interval in which v lies is sufficiently large.

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