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UNKNOTTING LINKS IN S^3 BY MAPS

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ABSTRACT. There exists a link $L_0 = L_{01} \cup L_{02}$ such that for no strongly 1-1 map f on L_0 is it true that $f(L_{01}), f(L_{02})$ are unknotted.

Let L be a (PL) link in the Euclidean 3-sphere S^3 (i.e., $L = \bigcup_{i=1}^n L_i$, where each L_i is a polygonal simple closed curve in S^3 and $L_i \cap L_j = \emptyset$, $i \neq j$). Call a continuous (PL) map $f\colon X \to X'$ strongly 1-1 on $Y \subset X$ if $f|Y$ is a homeomorphism onto $f(Y)$, $f(X - Y) \cap f(Y) = \emptyset$ and f is locally 1-1 at each point of Y. Gail Johnson asked whether or not it is always possible to find a map of S^3 onto itself which is strongly 1-1 on L and such that each $f(L_i)$, $i = 1, 2, \ldots, n$, is unknotted. If $n = 1$, the answer is yes (see [3]). In this paper we show the answer is no for $n \geqslant 2$; that is, we show in Theorem 1 that the link $L_0 = L_{01} \cup L_{02}$, illustrated in Figure 1, admits no strongly 1-1 map which unknots L_{01} and L_{02}.

Everything is assumed to be in the piecewise linear category. If $X \subset Y$, then $N(X)$ will denote a closed regular neighborhood of X in Y.

F. Gonzalez pointed out to the author that it is necessary to assume f is locally 1-1 at each point of L. That is, Gonzalez showed that given any link L there is a map $f\colon S^3 \to S^3$ such that $f|L$ is a homeomorphism onto $f(L)$, $f(S^3 - L) \cap f(L) = \emptyset$ and each $f(L_i)$ is unknotted. (To see this, map each L_i homeomorphically onto the circle $\{(x, y, i) | x^2 + y^2 = 1\}$. Map S^3-Int $N(L)$ to the origin. For each $x \in f(L_i)$, map the meridional disk of $N(L_i)$ that corresponds to $f^{-1}(x)$ to the straight line segment connecting x to the origin in the obvious way.) We will need the following lemma whose proof follows easily from the fact that if f is simplicial, then f is a homeomorphism on each simplex which meets L.

LEMMA 0. *Suppose f is strongly 1-1 on the link L. Then there exists a regular neighborhood of L, $N(L)$, and a map g such that g is strongly 1-1 on $N(L)$ and $g|L = f|L$.*

For Lemma 1 we need some special notation relative to Figures 1 and 2. In Figure 1, let T be S^3 minus an open regular neighborhood of the link $L_{01} \cup L_{02}$. Also, let B be the punctured 2-sphere with boundary $\bigcup_{i=1}^3 b_i$. In Figure 2, let C be a cube with two handles such that $N(L_{01}) \subset C$ and $C \subset T \cup N(L_{01})$. Also, let H be an annulus with boundary components h_1 and h_2 as illustrated in Figure 2, where $\operatorname{Bd} H \subset \operatorname{Bd} C$ and $\operatorname{Int} H \subset T - C$.

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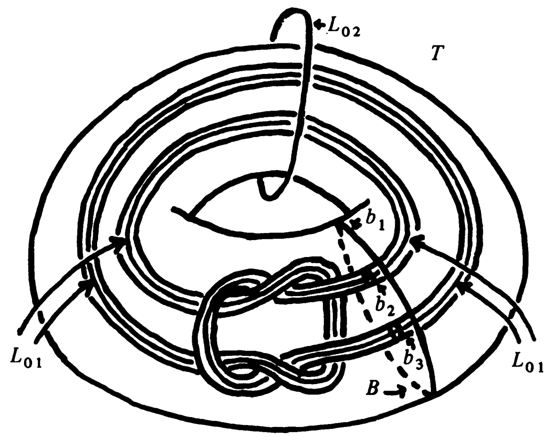

FIGURE 1

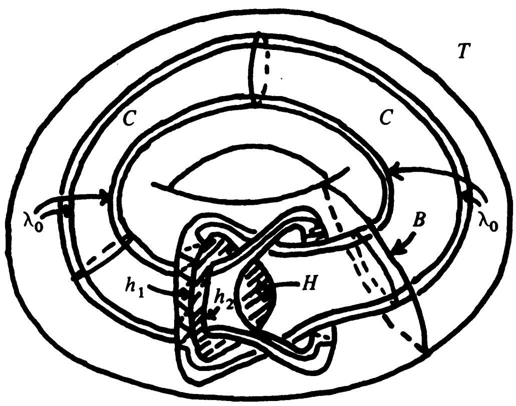

FIGURE 2

LEMMA 1. *If f: $S^3 \to S^3$ is strongly* 1-1 *on* $L_{01} \cup L_{02}$ *and* $f(L_{01}), f(L_{02})$ *are unknotted, then* $f(L_{01}), f(L_{02})$ *bound disjoint disks in* S^3.

PROOF. Suppose $f(L_{01})$ bounds the disk D_1 and that t, the number of points in $D_1 \cap f(L_{02})$, is smallest possible. By adjusting f to be transverse to D_1 and using Lemma 0, we suppose $f^{-1}(D_1) \cap T$ is an orientable surface $A = \bigcup_{i=0}^{q} A_i$ where the A_i's are the components of A, and A_0 contains the boundary component of A in $\operatorname{Bd} N(L_{01})$ parallel to L_{01} (assume

$$f^{-1}((D_1) \cap (N(L_{01}) \cup N(L_{02})))$$

consists of one annulus with one boundary component L_{01} and the other a longitude of $N(L_{01})$ and t meridional disks of $N(L_{02})$). Suppose A is compressible [2]. Then there exists a disk D with $\operatorname{Bd} D \subset \operatorname{Int} A$, $\operatorname{Int} D \subset \operatorname{Int} T - A$ and $\operatorname{Bd} D$ does not bound a disk in A. Either $f(\operatorname{Bd} D)$ is null homotopic in $D_1 - f(L_{02})$ or it is not. In the former case we may "cut" along D to form a new surface A (see [2] for a detailed description of this process). If $f(\operatorname{Bd} D)$

is not null homotopic in $D_1 - f(L_{02})$, then it would follow by Dehn's Lemma [6] and the Loop Theorem [7] that t was not minimal, contradiction. By repeating the above argument, we may suppose A is incompressible in T. Note that $f(A_0)$ must cover all of D_1 except for the interior of a regular neighborhood of $f(L_{01}) \cup (D_1 \cap f(L_{02}))$ in D_1, i.e., each A_i is a closed surface, $i > 0$.

Now suppose A_0 is boundary compressible [2]. Then there exists a disk D such that $\operatorname{Int} D \subset \operatorname{Int} T - A$; $\operatorname{Bd} D$ is composed of two arcs, one in $\operatorname{Bd} T$ (in particular $\operatorname{Bd} N(L_{02})$), the other in A_0; and the arc $D \cap A_0$ together with any arc in $\operatorname{Bd} A_0$ does not bound a disk in A_0. Note that $D \cap A_0$ cannot have its endpoints in the same component of $\operatorname{Bd} A_0$. Hence suppose $D \cap A_0$ connects different components of $\operatorname{Bd} A_0$. Then by Dehn's Lemma and the Loop Theorem we may find a real disk which makes $D_1 - (\operatorname{Int} N(f(L_{01}) \cup f(L_{02})))$ compressible in $S^3 - (\operatorname{Int} N(f(L_{01}) \cup f(L_{02})))$, and it follows that t was not minimal, contradiction. Hence we may suppose A is incompressible and boundary incompressible. Without loss of generality assume $A = A_0$.

Now suppose A is in general position relative to B and H (see Figures 1 and 2). Either there is an arc of $A \cap B$ which starts in b_2 and ends in b_3, or there is an arc starting in b_2 and ending in b_1 and an arc starting in b_3 and ending in b_1. In the former case it follows that $t = 0$, i.e., $D_1 \cap f(L_{02}) = \varnothing$. In the latter case we may adjust A in T so that $A \cap C$ (C is the cube with two handles illustrated in Figure 2) consists of an annulus E and s disks with boundaries $\lambda_1, \lambda_2, \ldots, \lambda_s$. The annulus E has one boundary component $\operatorname{Bd} A \cap (\operatorname{Bd} N(L_{01}))$, and the other is the λ_0 illustrated in Figure 2 ($\lambda_0 \subset \operatorname{Bd} C$ and λ_0 traverses each handle of C once and may twist $+n$ times around one handle and $-n$ times around the other handle). To see that such an intersection $A \cap C$ exists, think of C as very thin and the handles of C being close to and concentric with the portions of $N(L_{01})$ running through these two handles. It follows that there must be an arc θ of $A \cap H$ which starts and ends on h_2, the boundary component of H which separates $\operatorname{Bd} C$. Assume this arc does not separate any other components of $A \cap H$ in H (i.e., it is "innermost"). Now push this arc down and slightly into $\operatorname{Int} C$. If θ starts in λ_0 and ends in λ_j, $i > j > 0$, then it follows that we may reduce s by 2. If $i > 0, j = 0$, then we reduce s by 1. If θ starts and ends on λ_0, then $s = 0$ and the adjusted A does not intersect h_2. By repeating the above process we conclude that $A \cap h_2 = \varnothing$. It then follows that $A \cap B$ must contain an arc connecting b_2 to b_3, contradicting the initial assumption of this case. So $t = 0$ and Lemma 1 follows.

THEOREM 1. *There exists a two-component link L such that there is no map which is strongly* 1-1 *on L and unknots both components of L.*

PROOF. Suppose there exists a map f which is strongly 1-1 on $L_0 = L_{01} \cup L_{02}$. By Lemma 1, $f(L_{01}), f(L_{02})$ bound disjoint disks in S^3. Now, by following the techniques used in the proof of Lemma 1, there exist disjoint orientable surfaces F_1, F_2 such that $\operatorname{Bd} F_1 = L_{01}$ and $\operatorname{Bd} F_2 = L_{02}$, i.e., $L_{01} \cup L_{02}$ is a boundary link. But according to [1, p. 235] $L_{01} \cup L_{02}$ is not a boundary link, and the proof of Theorem 1 is complete.

In conclusion, we note that our result would have been much easier to obtain if we could apply the algebraic Theorem 1 of [4], i.e., we would be able to conclude that $b_1 \in \text{Kernel} f$ since $b_1 \in (\pi_1(T))_\omega$. Unfortunately the hypothesis of Theorem 1 of [4] is not satisfied since link groups of unsplittable links of more than one component are not free.

QUESTION. Is there an algebraic property of link groups which would give a theorem analogous to Theorem 1 of [4]? Note that Jaco and McMillan proved Theorem 1 of [5] much more easily by using their Theorem 1 of [4].

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