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HEEGAARD SPLITTINGS AND A THEOREM OF LIVESAY

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ABSTRACT. Let M be a nonorientable 3-manifold which is double covered by $S^2 \times I$. We give a short proof of the theorem of Livesay [1] that M is homeomorphic to $P^2 \times I$ (where P^2 denotes the projective plane).

0. Let M be a compact connected nonorientable 3-manifold with ∂M consisting of two copies of P^2 and $\pi_1(M) = Z_2$. Using the analysis of the Heegaard splittings of S^3 in [4], we prove the following

THEOREM [1]. *If the orientable two-fold cover of M is homeomorphic to $S^2 \times I$ then there is an annulus A embedded in M with $A \cap \partial M = \partial A$ consisting of two simple closed noncontractible curves, one in each component of ∂M.*

As in [1] the fact that M is homeomorphic to $P^2 \times I$ follows immediately. We divide the proof of this theorem into a number of steps, working throughout in the PL category.

1. Let $r\colon M \to P$ be a retraction such that r restricted to each component of ∂M is a homeomorphism. r can be chosen transverse to a simple closed noncontractible curve α in P. Then $r^{-1}(\alpha)$ is a compact 2-manifold embedded in M. Exactly as in [1] it follows that $r^{-1}(\alpha)$ contains a component K which is orientable, one-sided and has ∂K equal to two noncontractible simple closed curves, one in each component of ∂M.

Let $p\colon S^2 \times I \to M$ be the double covering and let $g\colon S^2 \times I \to S^2 \times I$ be the covering transformation. Let L denote $p^{-1}(K)$. If W is the closure of a component of $S^2 \times I - L$, then $W \cup gW = S^2 \times I$ and $W \cap gW = L$.

Clearly we can assume without loss of generality that K is incompressible, i.e., there is no disk D embedded in M with $D \cap K = \partial D$ and ∂D a noncontractible curve in K. Also it will be supposed that genus $K > 0$, i.e., K is not an annulus.

2. We show that K incompressible implies W is a handlebody. By [2] there are disjoint simple closed noncontractible curves $C_1, \ldots, C_m$ in ∂W such that the normal closure of the elements of $\pi_1(\partial W)$ given by joining each C_i to the base point along some path for $1 \leqslant i \leqslant m$ is Ker Φ (where $\Phi\colon \pi_1(\partial W) \to \pi_1(W)$ is induced by the inclusion map).

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By [3], $C_1, \ldots, C_m$ bound disks $D_1, \ldots, D_m$ in W which can be easily made disjoint from each other, with $D_i \cap \partial W = C_i$ for all i. Without loss of generality we can assume that $C_1, \ldots, C_m$ lie in L. Let $N(D_i)$ (or $N(L)$) be a small closed regular neighbourhood of D_i (or L) in W (or $S^2 \times I$) and let J denote the boundary of $gW \cup N(L) \cup N(D_1) \cup \cdots \cup N(D_m)$. Then J is mapped homeomorphically into M by p.

Suppose that J is not a union of 2-spheres. Let X, Y be the closures of the components of $M - p(J)$, with X homeomorphic to $W - \text{int}\, N(D_1) - \cdots - \text{int}\, N(D_m)$ and Y equal to $N(K) \cup pN(D_1) \cup \cdots \cup pN(D_m)$ (where $N(K) = pN(L)$). Then there is a simple closed noncontractible curve C in $p(J)$ which contracts in X or Y by [2]. As $C_1, \ldots, C_m$ give rise to elements of $\pi_1(\partial W)$ which normally generate Ker Φ, C must contract in Y.

Consequently by [3] there is a disk D embedded in Y with $D \cap \partial Y = \partial D = C$. Since K is incompressible, $D \cap K = \varnothing$ can be arranged. But $N(K) - K$ is homeomorphic to $L \times (0, 1]$ and so $Y - K$ is homotopically equivalent to $L \cup D_1 \cup \cdots \cup D_m$. This implies that C contracts in $L \cup D_1 \cup \cdots \cup D_m$ and therefore also in J, which is a contradiction.

This establishes that J is a union of 2-spheres and the fact that W is a handlebody follows directly.

3. Let genus L be denoted by n. By [4] since ∂W gives a Heegaard splitting of S^3 with two open 3-cells removed, there is a good system of meridian surfaces $D = D_1 \cup \cdots \cup D_n$ in W and a D-coordinate system $\bar{D} = \bar{D}_1 \cup \cdots \cup \bar{D}_n$ in gW. As defined in (2.1) of [4], this means that D, $\bar{D}$ are both collections of disjoint disks, such that $C_i \cap \bar{C}_j$ is a single transverse crossing point for $i = j$ and is empty for $i > j$, where $C_i = \partial D_i = D_i \cap \partial W$ and $\bar{C}_j = \partial \bar{D}_j = \bar{D}_j \cap \partial gW$ for all i, j. Without loss of generality we can assume that C_i, $\bar{C}_i$ are contained in L for all i.

Then also $g\bar{D}$ is a good system of meridian surfaces in W and gD is a $g\bar{D}$-coordinate system (with the ordering of the disks reversed). We want to separate the systems $(D, \bar{D})$ and $(g\bar{D}, gD)$ as in Lemma (2.5)(2) of [4]. Clearly we can assume that D and $g\bar{D}$ are transverse and $D \cap g\bar{D}$ consists of arcs only (any simple closed curves can be easily eliminated). Also it can be supposed that $D \cap g\bar{D} \cap gD = \varnothing$.

4. Here D, $\bar{D}$ (or $g\bar{D}$, gD) correspond to v, w (or x, y) in [4]. Let k be an arc of $D_i \cap g\bar{D}_j$ and let $N(k)$ be a small closed regular neighbourhood of k in W.

We can apply the procedure of Lemma (2.5)(2) to the system D, $\bar{D}$ and the arc k, producing a new system D', $\bar{D}'$. Next the same construction employed on $g\bar{D}'$, gD' (or $\bar{D}'$, D' with the reverse ordering of disks again) using the arc k (or gk) gives as outcome a system $g\bar{D}''$, gD'' (or $\bar{D}''$, D'').

It is easy to see that D'', $\bar{D}''$ is a good system of meridian surfaces for the splitting given by W'', gW'' and $L'' = W'' \cap gW''$, where $W'' = (W - \text{int}\, N(k)) \cup gN(k)$. Also $D'' \cap g\bar{D}''$ has at least one component less than $D \cap g\bar{D}$ and so after a finite number of steps we find that $D \cap g\bar{D} = \varnothing$ is achieved.

5. We can now utilize the argument in Satz (3.1) of [4]. As in (3.2) of [4], it follows that: *By modification of D only we reach that $gD \cap C_n$ contains at most one point* (and furthermore $D \cap g\bar{D} = \varnothing$ still holds).

The proof is by induction on the number of points in $gD \cap C_n$, where C_i and gC_j are assumed transverse for all i, j.

Case 1. *Some curve gC_j meets C_n in at least two points.*

By the method in [4], an arc k of C_n is found satisfying ∂k is contained in gC_j and int k is disjoint from $gD \cup \bar{D}$. Also if f and h are the arcs of gC_j with $\partial f = \partial h = \partial k$ then both $gC_j^0 = k \cup f$ and $gC_j^1 = k \cup h$ contract in gW. As $D \cap g\bar{D} = \varnothing$, k is also disjoint from $g\bar{D}$. Therefore $(C_j^0 \cup C_j^1) \cap \bar{D} = C_j \cap \bar{D}$ and we can suppose that $C_j^0 \cap \bar{C}_j = \varnothing$.

We replace C_j by C_j^1. Since (int k) $\cap \bar{D} = \varnothing$, C_j^1 bounds a disk D_j^1 in W with $D_j^1 \cap g\bar{D} = \varnothing$. If $j \neq n$ (or $j = n$) then $gC_j^1 \cap C_n$ (or $gC_n^1 \cap C_n^1$) has at least two points less than $gC_j \cap C_n$ (or $gC_n \cap C_n$), assuming the intersection is made transverse.

Case 2. *gC_j meets C_n in at most one point for all j, but $gD \cap C_n$ contains at least two points.*

As in [4] there is an arc k of C_n with $k \cap \bar{D} = \varnothing$ and $\partial k = k \cap gD = (k \cap gC_i) \cup (k \cap gC_j)$, for $i < j$ say. Let C_j' be the curve given by joining C_i and C_j along gk. Then $C_j' \cap \bar{D} = (C_i \cup C_j) \cap \bar{D}$ since $gD \cap \bar{D} = \varnothing$, and $gC_j' \cap \bar{D} = \varnothing$ because $k \cap \bar{D} = \varnothing$. Finally as $C_n \cap gC_n$ has an even number of points, $i, j \neq n$. Therefore replacing C_j by C_j', we decrease the number of points in $gD \cap C_n$ by one.

6. Remark. $gD \cap C_n = \varnothing$ implies that C_n contracts in gW as well as W and so C_n is null homologous in ∂W, which contradicts $C_n \cap \bar{C}_n$ is a single transverse crossing point.

So we can assume that $gC_j \cap C_n$ is a single transverse crossing point and $gC_i \cap C_n = \varnothing$ for all $i \neq j$, where $j \neq n$. In particular, $gC_n \cap C_n = \varnothing$. Let $W^* = (W - \text{int}\, N(D_n)) \cup gN(D_n)$ and let $L^* = W^* \cap gW^*$. Clearly L^* is a g-invariant surface in $S^2 \times I$.

Let $D^* = D - D_j - D_n$ and let $\bar{D}^* = \bar{D} - \bar{D}_j - \bar{D}_n$. Since $C_i \cap gC_n = \varnothing$ for all $i \neq j$, D^* is contained in W^* with $D^* \cap L^* = \partial D^*$. Also as $\bar{D} \cap gD = \varnothing$ and $\bar{C}_i \cap C_n = \varnothing$ for all $i < n$, it follows that $\bar{D}^*$ is included in gW^* and satisfies $\bar{D}^* \cap L^* = \partial \bar{D}^*$. Consequently D^* is a good system of meridian surfaces and $\bar{D}^*$ is a D^*-coordinate system for L^*.

7. Clearly the pair D^*, $\bar{D}^*$ found in §6 satisfies $D^* \cap g\bar{D}^* = \varnothing$. Let us denote $p(L^*)$ by K^*. Then K^* has genus one less than K and using the system D^*, $\bar{D}^*$ we can repeat the argument in §§5 and 6 above. Consequently after a finite number of steps the genus of K is reduced to zero and the theorem is proved.

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