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A SIMPLE PROOF OF A THEOREM OF CHACON

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ABSTRACT. A short and simple proof of a theorem of Chacon is presented by an application of a maximal inequality. A pointwise convergence theorem and the submartingale convergence theorem follow immediately from the theorem.

Here we present a short and simple proof of a theorem, due to Chacon [3], which implies a pointwise convergence theorem [1] and the submartingale convergence theorem [1], [2], [4].

THEOREM (CHACON). *Let* $\{X_n\}$ *be a sequence of integrable random variables such that* $\liminf_{n\to\infty} E(|X_n|) < \infty$. *Let*

$$X^* = \limsup_{n\to\infty} X_n, \quad X_* = \liminf_{n\to\infty} X_n,$$

and T be the collection of all bounded stopping times. Then

$$\limsup_{\tau,t\in T} E(X_\tau - X_t) \geqslant E(X^* - X_*). \tag{1}$$

Further, if $\sup_{\tau\in T} E(|X_\tau|) < \infty$, *then* X^* *and* X_* *are integrable.*

PROOF. By Lemma 1 of [1] and the Borel-Cantelli lemma, we can choose two strictly increasing sequences $\{\tau_k\}$ and $\{t_k\}$ of bounded stopping times such that $\lim_{k\to\infty} X_{\tau_k} = X^*$ almost surely and $\lim_{k\to\infty} X_{t_k} = X_*$ almost surely. Hence, the second assertion follows immediately from Fatou's lemma and we need only prove (1). To prove (1), it suffices to show that

$$\sup_{\tau,t\in T} E(X_\tau - X_t) \geqslant E(X^* - X_*). \tag{2}$$

It is also easy to see that, under the assumption of the theorem, if

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$\sup_{t\in T}E(|X_t|) = \infty$, then $\sup_{\tau,t\in T}E(X_\tau - X_t) = \infty$. Hence, we can, and do, assume that $\sup_{\tau\in T}E(|X_t|) < \infty$.

To prove (2), we need the following maximal inequality, which I learned from Chacon and Sucheston.

(3) $\lambda P\left(\left[\sup_n |X_n| \geqslant \lambda\right]\right) \leqslant \sup_{\tau\in T} E(|X_\tau|)$ for each positive constant λ.

To see (3), let M be a fixed positive integer and define a bounded stopping time τ by $\tau(w) = \inf\{n | 1 \leqslant n \leqslant M, |X_n(w)| \geqslant \lambda\}, \tau(w) = M + 1$ if no such n exists, $w \in \Omega$. Then

$$\lambda P\left(\left[\sup_{1\leqslant n\leqslant M} |X_n| \geqslant \lambda\right]\right) \leqslant E(|X_\tau|) \leqslant \sup_{t\in T} E(|X_t|).$$

(3) follows immediately on letting $M \to \infty$.

Now let λ be a positive constant, $\gamma(w) = \inf\{n | |X_n(w)| \geqslant \lambda\}, \gamma(w) = \infty$ if no such n exists, $w \in \Omega$. Let $A = [\gamma < \infty]$, $Y = \lambda\chi_{A^c} + |X_\gamma \chi_A|$, $Y_n = X_{n\wedge\gamma}$ for all $n \geqslant 1$, $Y^* = \lim\sup_{n\to\infty} Y_n$, and $Y_* = \lim\inf_{n\to\infty} Y_n$. By Lemma 1 of [1] and the Borel-Cantelli lemma, we can choose two strictly increasing sequences $\{\tau_k\}$ and $\{t_k\}$ of bounded stopping times such that $\lim_{k\to\infty} Y_{\tau_k} = Y^*$ almost surely and $\lim_{k\to\infty} Y_{t_k} = Y_*$ almost surely. Since $|Y_t| \leqslant Y$ for all $t \in T$ and $E(Y) \leqslant \lambda + \sup_{t\in T}E(|X_t|) < \infty$, by Lebesgue's dominated convergence theorem, $\lim_{k\to\infty} E(Y_{\tau_k} - Y_{t_k}) = E(Y^* - Y_*)$. So $\sup_{\tau,t\in T}E(Y_\tau - Y_t) \geqslant E(Y^* - Y_*)$. Since $\{Y_t | t \in T\} = \{X_{t\wedge\gamma} | t \in T\}$ is a subset of $\{X_t | t \in T\}$, $\sup_{\tau,t\in T}E(X_\tau - X_t) \geqslant E(Y^* - Y_*)$. By (3), (2) follows on letting $\lambda \to \infty$ (since X^* and X_* are integrable).

COROLLARY 1 (THEOREM 2 OF [1]). *Suppose that* $E(|X_n|) < \infty$ *for all* $n \geqslant 1$ *and* $\lim\inf_{n\to\infty} E(|X_n|) < \infty$. *Consider the following two statements.*

(a) *The generalized sequence* $\{E(X_\tau) | \tau \in T\}$ *is convergent.*

(b) X_n *converges almost surely to a finite limit.*

Then (a) *implies* (b).

COROLLARY 2 (THE SUBMARTINGALE CONVERGENCE THEOREM). *Suppose that* $\{X_n\}$ *is a sequence of* L_1*-bounded random variables adapted to the increasing sequence* $\{\mathcal{F}_n\}$ *of σ-fields. Suppose that* $E(X_{n+1} | \mathcal{F}_n) \geqslant X_n$ *a.s. for all* $n \geqslant 1$. *Then* X_n *converges almost surely to a finite limit.*

REMARK. The theorem and Corollary 1 also hold under any one of the following two conditions.

(i) $\sup_n E(X_n^+) < \infty$.

(ii) $\sup_n E(X_n^-) < \infty$.

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REFERENCES

1. D. G. Austin, G. A. Edgar and A. Ionescu Tulcea (1974), *Pointwise convergence in terms of expectations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **30**, 17–26. MR **50** #11402.

2. J. R. Baxter (1974), *Pointwise in terms of weak convergence*, Proc. Amer. Math. Soc. **46**, 395–398.

3. R. V. Chacon (1974), *A "stopped" proof of convergence*, Advances in Math. **14**, 365–368.

4. C. W. Lamb (1973), *A short proof of the martingale convergence theorem*, Proc. Amer. Math. Soc. **38**, 215–217. MR **48** #3119.

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