A SIMPLE PROOF OF A THEOREM OF CHACON

ROBERT CHEN

Abstract. A short and simple proof of a theorem of Chacon is presented by an application of a maximal inequality. A pointwise convergence theorem and the submartingale convergence theorem follow immediately from the theorem.

Here we present a short and simple proof of a theorem, due to Chacon [3], which implies a pointwise convergence theorem [1] and the submartingale convergence theorem [1], [2], [4].

Theorem (Chacon). Let \( \{X_n\} \) be a sequence of integrable random variables such that \( \lim \inf_{n \to \infty} E(|X_n|) < \infty \). Let

\[
X^* = \limsup_{n \to \infty} X_n, \quad X_* = \liminf_{n \to \infty} X_n,
\]

and \( T \) be the collection of all bounded stopping times. Then

\[
\limsup_{\tau, t \in T} E(X_{\tau} - X_{t}) \geq E(X^* - X_*).
\]

Further, if \( \sup_{\tau \in T} E(|X_{\tau}|) < \infty \), then \( X^* \) and \( X_* \) are integrable.

Proof. By Lemma 1 of [1] and the Borel-Cantelli lemma, we can choose two strictly increasing sequences \( \{\tau_k\} \) and \( \{t_k\} \) of bounded stopping times such that \( \lim_{k \to \infty} X_{\tau_k} = X^* \) almost surely and \( \lim_{k \to \infty} X_{t_k} = X_* \) almost surely. Hence, the second assertion follows immediately from Fatou’s lemma and we need only prove (1). To prove (1), it suffices to show that

\[
\sup_{\tau, t \in T} E(X_{\tau} - X_{t}) \geq E(X^* - X_*).
\]

It is also easy to see that, under the assumption of the theorem, if
sup_{t \in T} E(|X_t|) = \infty, then sup_{r,t \in T} E(X_r - X_t) = \infty. Hence, we can, and do, assume that sup_{t \in T} E(|X_t|) < \infty.

To prove (2), we need the following maximal inequality, which I learned from Chacon and Sucheston.

(3) \lambda P \left( \sup_n \left| X_n \right| \geq \lambda \right) \leq \sup_{t \in T} E(|X_t|) \quad \text{for each positive constant } \lambda.

To see (3), let M be a fixed positive integer and define a bounded stopping time \tau by \tau(w) = \inf\{n|1 \leq n \leq M, |X_n(w)| \geq \lambda\}, \tau(w) = M + 1 if no such n exists, w \in \Omega. Then

\lambda P \left( \sup_{1 \leq n \leq M} \left| X_n \right| \geq \lambda \right) \leq E(|X_\tau|) \leq \sup_{t \in T} E(|X_t|).

(3) follows immediately on letting M \to \infty.

Now let \lambda be a positive constant, \gamma(w) = \inf\{n|1 \leq n \leq M, |X_n(w)| \geq \lambda\}, \gamma(w) = \infty if no such n exists, w \in \Omega. Let A = [\gamma < \infty], Y = \lambda X_{\gamma} + |X_{\gamma} X_A|, Y_n = X_{n \wedge \gamma} for all n \geq 1, Y^* = \limsup_{n \to \infty} Y_n, and Y_* = \liminf_{n \to \infty} Y_n. By Lemma 1 of [1] and the Borel-Cantelli lemma, we can choose two strictly increasing sequences \{t_k\} and \{t_k\} of bounded stopping times such that \lim_{k \to \infty} Y_{t_k} = Y^* almost surely and \lim_{k \to \infty} Y_{t_k} = Y_* almost surely. Since |Y_{t_k}| \leq Y for all t \in T and E(Y) \leq \lambda + sup_{t \in T} E(|X_t|) < \infty, by Lebesgue's dominated convergence theorem, \lim_{k \to \infty} E(Y_{t_k} - Y_{t_k}) = E(Y^* - Y_*) \leq E(Y^* - Y_*). Since \{Y_{t_k}|t \in T\} = \{X_{t_k \wedge T}|t \in T\} is a subset of \{X_{t}|t \in T\}, sup_{t \in T} E(X_t - X_t) \geq E(Y^* - Y_*). By (3), (2) follows on letting \lambda \to \infty (since X^* and X_* are integrable).

**Corollary 1 (Theorem 2 of [1]).** Suppose that E(|X_n|) < \infty for all n \geq 1 and \liminf_{n \to \infty} E(|X_n|) < \infty. Consider the following two statements.

(a) The generalized sequence \{E(X_t)|t \in T\} is convergent.

(b) X_n converges almost surely to a finite limit.

Then (a) implies (b).

**Corollary 2 (The Submartingale Convergence Theorem).** Suppose that \{X_n\} is a sequence of L_1-bounded random variables adapted to the increasing sequence \{\mathcal{F}_n\} of \sigma-fields. Suppose that E(X_{n+1} | \mathcal{F}_n) \geq X_n a.s. for all n \geq 1. Then X_n converges almost surely to a finite limit.

**Remark.** The theorem and Corollary 1 also hold under any one of the following two conditions.

(i) sup_n E(X_n^+) < \infty.

(ii) sup_n E(X_n^-) < \infty.

**Acknowledgements.** I would like to thank Professors Chacon and Sucheston for their valuable suggestions and comments. I also want to thank the referee for his invaluable and useful suggestions.
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124