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ZERO SETS AND EXTENSIONS OF BOUNDED HOLOMORPHIC FUNCTIONS IN POLYDISCS

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ABSTRACT. A sufficient condition for a hypersurface in a polydisc U^n to be the zero set of an $H^\infty(U^n)$ function is proved. This strengthens a result of Zarantonello and generalizes a result of Rudin. Using this result and a result of Andreotti and Stoll, a partial extension of Alexander's theorem on extension of bounded holomorphic functions from a hypersurface of U^n to U^n is obtained. Finally, a generalization of Cima's extension theorem for H^p functions is given.

1. **Introduction.** Let U^n be the open unit polydisc in the space $\mathbf{C}^n$ of n complex variables. Let $N(U^n)$ and $H^p(U^n)$, $0 < p \leqslant \infty$, denote the Nevanlinna and Hardy classes respectively. (For definitions, see [6].) Rudin [6, Theorem 4.8.3] first gave a sufficient condition for the zero sets of $H^\infty(U^n)$. Later, Zarantonello [9] gave a sufficient condition for the zero sets of $N(U^n)$. In this paper, we show that Zarantonello's condition is also a sufficient condition for the zero sets of $H^\infty(U^n)$. This generalizes both the results of Rudin and of Zarantonello, and answers a question raised at the end of [9].

Next we consider the extension of bounded holomorphic functions from a hypersurface of U^n to U^n. This problem has been discussed by Alexander [2] and Andreotti and Stoll [3]. Using the above result, and a theorem of Andreotti and Stoll, we give a partial extension of Alexander's extension theorem.

Finally, a generalization of the extension theorem of Cima [4] is given in §5.

2. **Notations.** The following notations will be used. If $0 < r \leqslant 1$, then $U(r) = \{z \in \mathbf{C}\colon |z| < r\}$; as usual, we write U for $U(1)$. If $0 < r < s$, then $Q(r, s) = \{z \in \mathbf{C}\colon r < |z| < s\}$. The unit circle $\{z \in \mathbf{C}\colon |z| = 1\}$ is denoted by T and the unit n-torus by $T^n = T \times \cdots \times T$ (n copies). T^n is the distinguished boundary of U^n.

If Ω is an open subset of $\mathbf{C}^n$, then $H(\Omega)$ denotes the set of all holomorphic functions in Ω, and $H^\infty(\Omega)$ denotes the subset of all bounded ones. The zero set of $f \in H(\Omega)$ is $Z(f) = f^{-1}(0)$.

A subvariety E of U^n is said to satisfy Zarantonello's condition if

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there exists a constant $0 < r < 1$ and a continuous function $\eta = [r, 1) \to [r, 1)$ such that

$$|z_n| \leqslant \eta\left(\frac{|z_1| + \cdots + |z_{n-1}|}{n-1}\right) \tag{2.1}$$

for all $z = (z_1, \ldots, z_n) \in E \cap Q^n(R, 1)$.

3. **Zero sets of** $H^\infty(U^n)$. The following theorem gives a strengthening of the result of [9]. The proof is similar to that of [9].

THEOREM 3.1. *Suppose $n \geqslant 2$ and $f \in H(U^n)$. If $E = Z(f)$ satisfies Zarantonello's condition* (2.1), *then there exists an $F \in H^\infty(U^n)$ such that $f = Fe^h$ for some $h \in H(U^n)$.*

PROOF. Choose an $r \in (0, 1)$ and a continuous function $\eta: [r, 1) \to [r, 1)$ such that (2.1) is satisfied. Fix $r' \in (r, 1)$. Choose c such that

$$1 > c > c' = \sup\{\eta(x): r \leqslant x \leqslant 1 - (1 - r')/(n - 1)\}.$$

Let

$$V_i = U^{i-1} \times U(r') \times U^{n-i}, \qquad 1 \leqslant i \leqslant n - 1,$$
$$V_n = Q^{n-1}(r, 1) \times U.$$

Further, let

$$Q_i = Q^{i-1}(r, 1) \times Q(r, r') \times Q^{n-i-1}(r, 1) \times Q(c, 1), \qquad 1 \leqslant i \leqslant n - 1.$$

Note that

$$V_i \cap V_k = U^{i-1} \times U(r') \times U^{k-i-1} \times U(r') \times U^{n-k}, \qquad 1 \leqslant i < k \leqslant n - 1,$$

and

$$V_i \cap V_n = Q^{i-1}(r, 1) \times Q(r, r') \times Q^{n-i-1}(r, 1) \times U, \qquad 1 \leqslant i \leqslant n - 1,$$

are polydomains whose distinguished boundaries are products of the boundaries of the factors. In particular, the distinguished boundary of $V_i \cap V_n$ is contained in that of Q_i, $1 \leqslant i \leqslant n - 1$.

The polydomains $\{V_i: 1 \leqslant i \leqslant n\}$ form an open cover of U^n. They can be enlarged to form an open cover of $\overline{U}^n$ such that the intersection of the enlargement of V_i with U^n is V_i. We proceed to construct bounded Cousin data for the cover $\{V_i\}$ and then apply Stout's theorem [8].

Suppose $1 \leqslant i \leqslant n - 1$. If $(z_1, \ldots, z_{n-1}) \in Q^{i-1}(r, 1) \times Q(r, r') \times Q^{n-i-1}(r, 1)$ and $f(z_1, \ldots, z_{n-1}, z_n) = 0$, then

$$|z_n| \leqslant \eta\left(\frac{|z_1| + \cdots + |z_{n-1}|}{n-1}\right) \leqslant c' < c.$$

Hence $\operatorname{dist}(Z(f), Q_i) > 0$. It follows from Rudin's theorem [6, Theorem 4.8.3] (applied to the restriction of f to V_i) that there exists an $F_i \in H^\infty(V_i)$ such that $F_i f^{-1}$ is an invertible element of $H(V_i)$, and F_i^{-1} is bounded in Q_i.

Next, we show that the same is true in V_n. Fix $z' \in Q^{n-1}(r, 1)$. Then $f(z', \cdot)$ has finitely many zeros in U, by the hypothesis on f. Let these zeros be $\alpha_1(z'), \ldots, \alpha_k(z')$, listed according to multiplicities. Put

$$F_n(z) = \prod_1^k (z_n - \alpha_i(z')), \qquad z = (z', z_n) \in V_n. \tag{3.1}$$

Then k is independent of z' and $F_n \in H(V_n)$ (see [9, p. 312]). Clearly, F_n is bounded and has the same zeros as f in V_n. Since $\operatorname{dist}(Z(f), Q_i) \geqslant c - c' > 0$, it follows that $|F_n| \geqslant (c - c')^k > 0$ in Q_i. Hence F_n^{-1} is bounded in Q_i, $1 \leqslant i \leqslant n - 1$.

Since, for all i, $F_i f^{-1}$ is a zero-free holomorphic function in V_i, so is $F_i F_k^{-1}$ in $V_i \cap V_k$, for all i, k. We claim that $F_i F_k^{-1}$ is bounded in $V_i \cap V_k$, $1 \leqslant i$, $k \leqslant n$.

Suppose $1 \leqslant i < k \leqslant n - 1$. Then $F_i F_k^{-1}$ is holomorphic in $V_i \cap V_k$ and is bounded in Q_k. The distinguished boundary of $V_i \cap V_k$ is contained in $\bar{Q}_k$. Hence, by the maximum modulus theorem, $F_i F_k^{-1}$ is bounded in $V_i \cap V_k$. Similarly, $F_k F_i^{-1}$ is bounded in $V_i \cap V_k$.

Suppose $1 \leqslant i \leqslant n - 1$. Then $F_i F_n^{-1}$ is holomorphic in $V_i \cap V_n$ and is bounded in Q_i. Since the distinguished boundary of $V_i \cap V_n$ is contained in that of Q_i, the maximum modulus theorem again shows that $F_i F_n^{-1}$ is bounded in $V_i \cap V_n$. Similarly, $F_n F_i^{-1}$ is bounded in $V_i \cap V_n$.

Hence, for all i, k, $F_i F_k^{-1}$ is an invertible element of $H^\infty(V_i \cap V_k)$. By Stout's theorem [8], there exists an $F \in H^\infty(U^n)$ such that FF_i^{-1} is an invertible element of $H^\infty(V_i)$, $1 \leqslant i \leqslant n$. Since $F_i f^{-1}$ is an invertible element of $H(V_i)$, it follows that Ff^{-1} is zero-free in V_i, $1 \leqslant i \leqslant n$. Since $\{V_i : 1 \leqslant i \leqslant n\}$ covers U^n, Ff^{-1} is zero-free in U^n and so there exists an $h \in H(U^n)$ such that $f = Fe^h$.

Remark. For later applications, we note that for each i, there exists $\psi_i \in H^\infty(V_i)$ such that $\psi_i^{-1} \in H^\infty(V_i)$ and $F = F_i \psi_i$. Hence F^{-1} is bounded in Q_i, $1 \leqslant i \leqslant n - 1$.

4. **Extensions of bounded holomorphic functions.** Using the results of §3 and of Andreotti and Stoll [3], we can now give a partial extension of the result of Alexander [2].

THEOREM 4.1. *Let E be a subvariety of U^n, $n \geqslant 2$, of pure dimension $n - 1$, satisfying condition* (2.1) *and the following condition of Alexander*:

(4.1) *there exists a $\delta > 0$ such that if r is as in* (2.1), $1 \leqslant i \leqslant n$, (z', α, z'') *and* $(z', \beta, z'') \in E \cap [Q^{i-1}(r, 1) \times U \times Q^{n-i}(r, 1)]$, *and* $a \neq b$, *then* $|\alpha - \beta| \geqslant \delta$.

Then for all bounded holomorphic functions g on E, there exists a bounded holomorphic function G in U^n such that $G = g$ on E.

PROOF. Fix $r' \in (r, 1)$. Let c, V_i, Q_i be as defined in §3. It was shown in [2] that there exists $f \in H(U^n)$ such that $E = Z(f)$ and $\partial f / \partial z_i \neq 0$ on $E \cap$

$[Q^{i-1}(r, 1) \times U \times Q^{n-i}(r, 1)]$, $1 \leqslant i \leqslant n$. By Theorem 3.1, there exists $F \in H^\infty(U^n)$ such that $f = Fe^u$ for some $u \in H(U^n)$. Hence $\partial F/\partial z_i \neq 0$ on $E \cap [Q^{i-1}(r, 1) \times U \times Q^{n-i}(r, 1)]$, $1 \leqslant i \leqslant n$. By the remark at the end of §3, there exists $\psi \in H^\infty(V_n)$ such that $\psi^{-1} \in H^\infty(V_n)$ and $F = F_n\psi$. By condition (4.1) and definition (3.1) of F_n, it follows that $|\partial F_n/\partial z_n|$ is bounded from 0 on $E \cap V_n$. Since $\partial F/\partial z_n = \psi \partial F_n/\partial z_n$ on $E \cap V_n$, it follows that there exists $\varepsilon > 0$ such that $|\partial F/\partial z_n| \geqslant \varepsilon$ on $E \cap V_n$.

Let $g \in H^\infty(E)$. For $1 \leqslant i \leqslant n-1$, it follows from Alexander's theorem [2] that there exists $g_i \in H^\infty(V_i)$ such that $g = g_i$ on $E \cap V_i$. We show that the same is true in V_n.

By Cartan's theorem (see [6, Theorem 7.1.2]) there exists $\phi \in H(U^n)$ such that $\phi = g$ on $E \cap V_n$. Let $(z', z_n) \in V_n$, and let

$$s = \eta\left(\frac{|z_1| + \cdots + |z_{n-1}|}{n-1}\right).$$

Choose a positively oriented circle γ with center 0, lying in $Q(s, 1)$ and enclosing z_n. Put

$$h(z', z_n) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z', \zeta)}{F(z', \zeta)}\,\frac{d\zeta}{\zeta - z_n}.$$

Then h is independent of the choice of γ and $h \in H(V_n)$. Let $g_n = \phi - hF$. Then $g_n = g$ on $E \cap V_n$. We claim that $g_n \in H^\infty(V_n)$. Let $\gamma_1, \ldots, \gamma_k$ be small circles about the zeros $\alpha_1(z'), \ldots, \alpha_k(z')$ of $F(z', \cdot)$. Then by the computation given in [2, p. 488],

$$(\phi - hF)(z', z_n) = \sum_{j=1}^{k} \frac{g(z', \alpha_j(z'))}{(\partial F/\partial z_n)(z', \alpha_j(z'))} \cdot \frac{F(z', z_n)}{z_n - \alpha_j(z')}.$$

Since $F = F_n\psi$, each $F(z', z_n)/(z_n - \alpha_j(z'))$ is bounded on V_n. Since $|\partial F/\partial z_n| \geqslant \varepsilon$ on $E \cap V_n$, and each $(z', \alpha_j(z')) \in E \cap V_n$, it follows that $g_n = \phi - hF$ is bounded on V_n.

For $1 \leqslant i < k \leqslant n$, and $z = (z', z_n) \in V_i \cap V_k$, put

$$a_{ik}(z) = \frac{1}{2\pi i}\int_\gamma \frac{g_i(z', \zeta) - g_k(z', \zeta)}{F(z', \zeta)}\,\frac{d\zeta}{\zeta - z_n},$$

where γ is a positively oriented circle with center 0 and radius $> \max(c, |z_n|)$. Then a_{ik} is independent of the choice of γ and $a_{ik} \in H(V_i \cap V_k)$.

Suppose $1 \leqslant i < k \leqslant n-1$. Since $F = 0$ on E and $\partial F/\partial z_n \neq 0$ on $E \cap V_n$, $F(z', \cdot)$ has simple zeros. Since $g_i - g_k = 0$ on $E \cap V_i \cap V_k$, it follows that $(g_i - g_k)F^{-1}$ is holomorphic in $V_i \cap V_k \cap V_n$. Therefore, by Cauchy's integral formula, $a_{ik} = (g_i - g_k)F^{-1}$ in $V_i \cap V_k \cap V_n$. Since this is an open subset of $V_i \cap V_k$ which is connected, we must have

$$g_i - g_k = a_{ik}F \quad \text{in } V_i \cap V_k.$$

Since F^{-1} is bounded in $\overline{Q}_i$, and the distinguished boundary of $V_i \cap V_k$ is contained in $\overline{Q}_i$, $a_{ik} \in H^\infty(V_i \cap V_k)$ by the maximum modulus theorem.

Suppose $1 \leqslant i \leqslant n-1$. Then by the reasons given above, $(g_i - g_n)F^{-1}$ is holomorphic in $V_i \cap V_n$. So by Cauchy's integral formula,

$$g_i - g_n = a_{in}F \quad \text{in } V_i \cap V_n.$$

Since F^{-1} is bounded on Q_i, and the distinguished boundary of $V_i \cap V_n$ is contained in $\bar{Q}_i$, it follows by the maximum modulus theorem that $a_{in} \in H^\infty(V_i \cap V_n)$.

Now we conclude by Theorem 2.8 of [3] that there exists a $G \in H^\infty(U^n)$ such that $G = g$ on E.

REMARK. Alexander has shown that if E satisfies Rudin's condition that $\operatorname{dist}(E, T^n) > 0$, together with condition (4.1), then there exists a bounded linear operator $T: H^\infty(E) \to H^\infty(U^n)$ such that $Tf = f$ on E.

It follows quite easily from the open mapping theorem that under the hypothesis of Theorem 4.1, there exists a constant M such that every $f \in H^\infty(E)$ has an extension $F \in H^\infty(U^n)$ satisfying $\|F\|_{U^n} \leqslant M\|f\|_E$ (see [10, p. 517]). However, we do not know if the extension F can be chosen to depend linearly on f.

5. **Removable singularities.** Let E be a closed subset of U^n. For $0 < p < \infty$, we say that $f \in H^p(U^n - E)$ if f is holomorphic in $U^n - E$ and $|f|^p$ has an n-harmonic majorant in $U^n - E$. If E is empty, this condition is equivalent to the usual one, namely,

$$\sup_{0<r<1} \int_{T^n} |f(rw)|^p \, dm(w) < \infty,$$

where m is the normalized Haar measure on T^n. (See [6, Chapter 3].) We wish to consider whether E is a set of removable singularities of f.

For $n = 1$, Parreau [5, p. 182] has proved that if E has logarithmic capacity zero, then every $f \in H^p(U - E)$ can be extended to an $F \in H^p(U)$. For $n > 1$, it is a result of Shiffman [7, Lemma 3] that every set E with $(2n-1)$-dimensional Hausdorff measure zero is removable for $f \in H^\infty(U^n)$. If $1 \leqslant p < \infty$, Cima [4] has shown that E is removable if E is a hypersurface of U^n satisfying Rudin's condition: $\operatorname{dist}(E, T^n) > 0$. We show that in Cima's result, Rudin's condition can be replaced by Zarantonello's. Furthermore, we only require $|f|^p$ to have an n-harmonic majorant in $U^n - E$ instead of an RP-majorant as in [4].

THEOREM 5.1. *Let $n \geqslant 2$, $0 < p < \infty$. Suppose E is a subvariety of U^n of pure dimension $n-1$ satisfying condition* (2.1). *If $f \in H^p(U^n - E)$, then there exists an $F \in H^p(U^n)$ such that $F = f$ on $U^n - E$.*

PROOF. Since the case $\lim_{s \to 1} \eta(s) < 1$ is covered by [4], we may assume that $\lim_{s \to 1} \eta(s) = 1$.

Suppose $f \in H^p(U^n - E)$. We show first that f extends to a holomorphic function in U^n. By Theorem 2.1, there exists $g \in H^\infty(U^n)$ such that $E = Z(g)$. Let $a = (a_1, \ldots, a_n) \in E$. If $g(z', a_n) \not\equiv 0$ as a function of z', then by the proof given in [4, p. 531], f extends to a holomorphic function in U^n. If

$g(z', a_n) \equiv 0$, then there exist a positive integer α and a function $g_1 \in H(U^n)$ such that $g(z) = (z_n - a_n)^\alpha g_1(z)$, where $g_1(z', a_n) \not\equiv 0$. Hence $Z(g) = \{z \in U^n: z_n = a_n\} \cup Z(g_1)$. Since by the proof given in [4, p. 531], f extends holomorphically over both $\{z \in U^n: z_n = a_n\}$ and $Z(g_1)$, it follows that there exists $F \in H(U^n)$ such that $F = f$ on $U^n - E$.

To show that $F \in H^p(U^n)$, we note first that by the n-subharmonicity of $|F|^p$,

$$\sup_{0<r<1} \int_{T^n} |F(rw)|^p \, dm(w) = \sup_{0<s<1} \int_{T^n} |F(sw', tw_n)|^p \, dm(w),$$

where $t = \frac{1}{2}(1 + \eta(s))$, $w = (w', w_n) = (w_1, \ldots, w_{n-1}, w_n)$. By hypothesis, there exists an n-harmonic function u in $U^n - E$ such that $|F|^p \leqslant u$ in $U^n - E$. Hence

$$\int_{T^n} |F(sw', tw_n)|^p \, dm(w) \leqslant \int_{T^n} u(sw', tw_n) \, dm(w). \tag{5.1}$$

It is therefore sufficient to show that the last integral is bounded as $s \to 1$.

For $r' \in (r, 1)$, let $r_1 = \max\{\eta(x): r \leqslant x \leqslant r'\}$. Then u is n-harmonic in the polyannulus $Q^{n-1}(r, r') \times Q(r_1, 1)$. If $r_1 < t < 1$, then by a well-known result (see e.g. [1, Chapter 5]), we have

$$\int_T u(sw', tw_n) \, dm(w_n) = u_1(sw') \log t + v_1(sw')$$

where u_1 and v_1 are $(n-1)$ harmonic in $Q^{n-1}(r, r')$. Repeated integration gives

$$\int_{T^n} u(sw', tw_n) \, dm(w) = \sum_{i=0}^{n-1} (\alpha_i \log t + \beta_i)(\log s)^i$$

where α_i, β_i are constants. Since both s and t are bounded from 0 and ∞, it follows that the right side of (5.1) is bounded as $s \to 1$. This completes the proof.

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