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PRINCIPAL IDEALS IN F-ALGEBRAS

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ABSTRACT. This paper is concerned with generalizations to F-algebras of theorems which Gleason has proved for finitely generated maximal ideals in Banach algebras. Let A be a uniform commutative F-algebra with identity such that Spec (A) is locally compact; let x be a nonisolated point of Spec (A), and let $\ker(x)$ denote the maximal ideal of all elements of A which vanish at x. In this paper it is shown that: If f is an element of A vanishing only at x, then the principal ideal Af generated by f is closed in A. If the polynomials in the element f are dense in A and if $\ker(x)$ is finitely generated, then there exists an open set U containing x such that $\ker(y)$ is generated by $f - f(y)$ for all y in U. An example is given which shows that if A is not uniform, the conclusion of the last result may not be true. In fact, the example shows that it is possible to have a nonisolated finitely generated maximal ideal in the algebra. A second example shows that in a uniform F-algebra with locally compact spectrum, $\ker(x)$ can be generated by an element f such that $f - f(y)$ generates no other $\ker(y)$ even when the $\ker(y)$ are principal.

Introduction. The results in this paper generalize to F-algebras results which are known for Banach algebras (see Theorem 2.1(ii) and Theorem 2.2 of [4]). Although the results stated are only for principal maximal ideals, they should point out some of the difficulties in general for finitely generated maximal ideals.

Suppose that B is a commutative Banach algebra with identity. Gleason [4] proved the following theorems dealing with the generators of an algebraically finitely generated maximal ideal.

(G1) *If I is a maximal ideal of B which is generated by $g_1, \ldots, g_n$, then there exists a neighborhood U of I in the maximal ideal space of B such that each maximal ideal M in U is generated by $g_1 - \hat{g}_1(M), \ldots, g_n - \hat{g}_n(M)$.*

(G2) *If the subalgebra generated by $1, z_1, \ldots, z_k$ is dense in B and if I is a finitely generated maximal ideal in B, then I is generated by $z_1 - \hat{z}_1(I), \ldots, z_k - \hat{z}_k(I)$.*

One immediate consequence of these theorems is that in a commutative Banach algebra with identity, the set of finitely generated maximal ideals is an open set. We will see with an example that this need not be true for F-algebras. Furthermore, we will show that when one considers F-algebras the conclusion

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of (G1) will depend on which generators one takes. The positive results given for F-algebras deal with the case in (G2) where $k = 1$.

Principal ideals. Throughout this paper A will denote a uniform commutative F-algebra with identity such that Spec (A) is locally compact. A may be regarded as a (compact-open) closed subalgebra of the algebra of all continuous complex-valued functions on Spec (A). A subset K of Spec (A) is A-convex if for each x in Spec $(A) - K$, there is an element f of A such that $|f(x)| > \sup\{|f(y)|: y \in K\}$. Fix an increasing sequence $\{X_n\}$ of compact A-convex subsets of Spec (A) such that each compact subset of Spec (A) is contained in some X_n. Let A be the completion with respect to the supremum norm on X_n of the algebra $A|X_n$ of restrictions of elements of A to X_n. Then $A = \lim\operatorname{inv} A_n$ and $X_n = \operatorname{Spec}(A_n)$.

Suppose that x is a nonisolated point in Spec (A) and that the element f in A vanishes only at x. Define a mapping T: $g \to gf$ of A into A. Since x is not isolated, T is one-to-one. The range of T is Af, the principal ideal generated by f. If T is not a homeomorphism of A into A, then f is called a *strong topological divisor of zero* (see [5, p. 47]). But then Af is a closed ideal if and only if T is a homeomorphism. Hence, Af is a closed ideal if and only if f is not a strong topological divisor of zero.

We now show that if the maximal ideal $\ker(x) = \{g \in A: g(x) = 0\}$ is finitely generated as an ideal, then the principal ideal Af is closed. Notice that we do not assume that Af is the maximal ideal $\ker(x)$.

If $\ker(x)$ is finitely generated, then x is not in the Shilov boundary of A (see Theorem 4.11 of [2]). Consequently, for large n, x is not in ∂A_n, the Shilov boundary of A_n. Thus, by Lemma 2.2 of [3], if $\ker(x)$ is finitely generated, then f is not a strong topological divisor of zero. We have proved the following theorem.

THEOREM 1. *Suppose that x is a nonisolated point of* Spec (A) *such that* $\ker(x)$ *is finitely generated. If f is an element of A vanishing only at x, then the principal ideal Af generated by f is closed.*

As an immediate consequence of this theorem we get a result similar to Gleason's theorem (G2).

COROLLARY 2. *Suppose that the polynomials in the element g are dense in A and that x is a nonisolated point in* Spec (A) *such that* $\ker(x)$ *is finitely generated. Then* $\ker(x)$ *is principal and is generated by $g - g(x)$.*

PROOF. By replacing g by $g - g(x)$ we may assume that $g(x) = 0$. Since, with this assumption, g vanishes only at x, the principal ideal Ag generated by g is closed. But it is easy to see that Ag is dense in $\ker(x)$. Consequently $\ker(x) = Ag$.

COROLLARY 3. *If* $\ker(x) = Af$, *then the principal ideals Af^n are closed for $n \geqslant 1$.*

PROOF. Because $\ker(x) = Af$, f vanishes only at x and consequently f^n also vanishes only at x.

The rest of this paper deals with the problem of whether a generator of a principal maximal ideal must in the sense of (G1) generate nearby maximal ideals. We first give an example of an F-algebra in which this is not true. See [1] for details.

EXAMPLE 4. For all positive integers k and n, let

$$K_n = [-n, n],\ I_n = (-1/n, 1/n), \text{ and } I_{n,k} = [-1/n + 1/nk, 1/n - 1/nk].$$

Let D be the algebra of all continuous, complex-valued functions on the real line R which are n-times continuously differentiable on I_n for each n. For a compact set K in R, $\|\cdot\|_K$ will denote the supremum seminorm, and for positive integers n and j, $\|\cdot\|_{n,j}$ will denote the seminorm on D given by $\|f\|_{n,j} = \sum_{i=0}^{n} (1/i!)\|f^{(i)}\|_{I_{n,j}}$. With the topology on D induced by these seminorms, D is a semisimple, regular, commutative F-algebra with identity. Furthermore, D contains $C^\infty(R)$, the polynomials in the coordinate function are dense in D, and $\operatorname{Spec}(D) = R$. Finally, $\ker(0)$ is principal and is generated by the coordinate function. No other maximal ideal is finitely generated.

Notice that the algebra D is not a uniform algebra. We return to the uniform algebra A.

LEMMA 5. *If* $\ker(x)$ *is principal, then there exists an open set* U *containing* x *and an integer* n_0 *such that* $U \cap \partial A_n = \varnothing$ *for* $n \geqslant n_0$.

PROOF. Suppose that $\ker(x) = Af$. By Theorem 2.6 of [3], there exists an open set V containing x such that f: $V \to f(V)$ is a homeomorphism onto an open disc in C and $g \circ f^{-1}$ is analytic on $f(V)$ for all g in A. Let U be an open set containing x such that $\overline{U}$ is compact, $\overline{U} \subset V$, and $f(\overline{U})$ is a closed disc; let n_0 be large enough so that $\overline{U} \subset X_n$ for all $n \geqslant n_0$. If $n \geqslant n_0$ and $g \in A_n$, then $(g|\overline{U}) \circ f^{-1}$ is in the disc algebra on the disc $f(\overline{U})$. From this the conclusion follows.

LEMMA 6. *If* $\ker(x) = Af$ *and there exists an open set* V *containing* x *such that* hull $(f - f(y)) = \{y\}$ *for each* y *in* V, *then there exists an open set* U *containing* x *such that the principal ideals* $A(f - f(y))$ *are closed for each* y *in* U.

PROOF. This follows directly from the previous lemma and Lemma 2.2 of [3].

What we might like to prove for uniform F-algebras is: "If $\ker(x) = Af$, then there exists a neighborhood U containing x such that if $y \in U$, then $\ker(y) = A(f - f(y))$." For such a neighborhood U, it would necessarily be true that if $y \in U$, then hull $(f - f(y)) = \{y\}$. But the following example shows that this can fail to be true.

EXAMPLE 7. Let A be the algebra of all continuous complex-valued functions on the complex plane which are analytic on the set $|z| < 1$. Define f by $f(z) = z$ if $|z| \leqslant 1$ and $f(z) = 1/\bar{z}$ if $|z| > 1$. Then $\ker(0) = Af$, but if

$z_0 \neq 0$, then hull $(f - f(z_0)) = \{z_0, 1/\bar{z}_0\}$. Of course, there are better behaved generators of $\ker(0)$, namely z. Notice that this behavior cannot happen in a Banach algebra by Gleason's results. The following theorem follows easily from our earlier results.

THEOREM 8. *If the polynomials in the element f are dense in A, and if* $\ker(x)$ *is finitely generated, then there exists an open set U containing x consisting of principal maximal ideals*; *in fact, if $y \in U$, then* $\ker(y)$ *is generated by* $f - f(y)$.

Example 4 shows that the hypothesis of Theorem 8 that A be uniform is necessary.

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