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UNIQUENESS IN THE SCHAUDER FIXED POINT THEOREM[1]

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ABSTRACT. A condition is given which guarantees the uniqueness of the fixed point in the Brouwer and Schauder fixed point theorems. The result is applied to a nonlinear boundary value problem in physiology.

1. Let X be a real Banach space with a bounded convex open subset D, and let $F : \overline{D} \to \overline{D}$ be a continuous function which is also assumed to be compact if X is infinite dimensional. The Brouwer fixed point theorem (Schauder theorem if X is infinite dimensional) gives a point $x \in \overline{D}$ such that $x = F(x)$. Under the assumption that F is differentiable, we give a simple condition which guarantees that the fixed point x is unique. The proof is an application of degree theory. We phrase the argument for the infinite dimensional case; the reader who is interested only in the finite dimensional case may omit the compactness hypothesis.

In the last section, the result is applied to a nonlinear boundary value problem arising in physiology.

2. Suppose that $F : \overline{D} \to \overline{D}$ is compact and continuously Fréchet differentiable in D. Then [4, Lemma 4.1] $F'(x)$ is a compact linear operator on X for each $x \in D$. Our uniqueness result is

THEOREM. *Let $F : \overline{D} \to \overline{D}$ be a compact continuous map which is continuously Fréchet differentiable on D. Suppose that* (a) *for each $x \in D$, 1 is not an eigenvalue of $F'(x)$, and* (b) *for each $x \in \partial D$, $x \neq F(x)$. Then F has a unique fixed point.*

In order to prove the Theorem we require a lemma. For the Lemma, recall that for any compact linear operator A the spectrum, $\sigma(A)$, of A consists of a countable number of points having 0 as the only possible limit point. Each nonzero $\lambda \in \sigma(A)$ is an eigenvalue of A. For such a λ, the null space of $(A - \lambda I)^k$ is, for all k sufficiently large, independent of k. The dimension of this null space is called the algebraic multiplicity of the eigenvalue λ.

LEMMA. *Let A be a compact linear operator on a real Banach space X. Suppose $1 \notin \sigma(A)$, and let $\beta(A)$ denote the sum of the algebraic multiplicities of all $\lambda \in \sigma(A)$ with λ real and $\lambda > 1$. Then there is an $\varepsilon > 0$ such that if B is a compact linear operator on X and $\|A - B\| < \varepsilon$, then $(-1)^{\beta(A)} = (-1)^{\beta(B)}$.*

PROOF OF LEMMA. Let B be a compact operator with $\|A - B\| < \varepsilon$, where ε will be determined in the course of the proof. Letting $\varepsilon < 1$, we see that

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$\|B\| < M = \|A\| + 1$. Let Q be an open rectangle in the λ plane with two sides on $\operatorname{Re} \lambda = 1$ and $\operatorname{Re} \lambda = M$, and the other two sides on $\operatorname{Im} \lambda = \pm a$. Let Γ denote the boundary of Q. We pick $a > 0$ so small that $\sigma(A) \cap \overline{Q}$ consists only of real eigenvalues. Then $\Gamma \cap \sigma(A) = \varnothing$. For ε sufficiently small, we have $\Gamma \cap \sigma(B) = \varnothing$ [2, p. 213]. Let $R_A(\lambda) = (\lambda I - A)^{-1}$ be the resolvant operator of A, and set

$$P_A = \int_\Gamma (\lambda I - A)^{-1}\, d\lambda.$$

Then P_A is a projection whose range is the union of the eigenspaces of A corresponding to the eigenvalues $\lambda \in Q \cap \sigma(A)$ [2, p. 178]. Thus, setting $d(P_A) =$ the dimension of the range of P_A, we find that $d(P_A)$ is the sum of the algebraic multiplicities of the eigenvalues $\lambda \in \sigma(A) \cap Q$. Defining P_B in a similar way, we have a similar result for $d(P_B)$. For ε sufficiently small, we have $\|P_A - P_B\| < 1$, so $d(P_A) = d(P_B)$ [2, p. 33]. Since $\sigma(A) \cap Q$ consists only of real eigenvalues, and since $\lambda \in \sigma(A)$ satisfies $|\lambda| < M$, we have $d(P_A) = \beta(A)$. If $\lambda \in \sigma(B)$ is complex, then $\bar{\lambda} \in \sigma(B)$, and λ and $\bar{\lambda}$ have the same algebraic multiplicity. Hence $d(P_B) = \beta(B) +$ even number. Combining these facts, we find that $\beta(A)$ and $\beta(B)$ have the same parity, which proves the lemma.

Proof of Theorem. We first show that F has a finite number of fixed points. For supposing otherwise, let $x^k = F(x^k)$, $k = 1, 2, \ldots,$ be a sequence of fixed points. Since F is compact we may, after picking a subsequence, suppose that $x^k \to x \in \overline{D}$, $F(x) = x$. Hence $x \in D$. By the spectral condition, $I - F'(x)$ has a bounded inverse, so from the inverse function theorem [5, Theorem 1.20], $I - F(x)$ is (1-1) in a neighborhood of x, contradicting $(I - F)(x^j) = 0$. Let $x^1, \ldots, x^N$ denote the fixed points of F. Let U_j be a neighborhood of x^j such that the closed sets $\overline{U}_j$ are pairwise disjoint and $\overline{U}_j \subset D$. Let $K = \overline{D} \setminus \{\bigcup_1^N U_j\}$, so K is a closed subset of $\overline{D}$ which does not contain a fixed point of F. Then the quantities

$$\deg(0, I - F, D) \quad \text{and} \quad \deg(0, I - F, U_j)$$

are well defined, and from the excision and additive properties of the degree [5, Proposition 3.37], we have

$$\deg(0, I - F, D) = \deg(0, I - F, D \setminus K) = \sum_j \deg(0, I - F, U_j).$$

Without loss of generality we may suppose $0 \in D$. Then $H(x, t) = x - tF(x)$, $0 \leqslant t \leqslant 1$, defines a homotopy of F with the identity function I. Since $0 \in D$, $H(x, t) \neq 0$ for $x \in \partial D$, for otherwise $x = tF(x) + (1 - t)0 \in D$, which is a contradiction. Hence $\deg(0, I - tF, D)$ is defined, so using this homotopy,

$$\deg(0, I - F, D) = \deg(0, I, D) = 1.$$

From [4, Theorem 4.7],

$$\deg(I - F, 0, U_j) = (-1)^{\beta(x_j)}$$

where, in the notation of the Lemma, $\beta(x) = \beta(F'(x))$. Since $F'(x)$ is continuous, we see from the Lemma that the parity of $\beta(x)$ is constant for $x \in D$. Hence $1 = \pm N$, so $N = 1$ and the fixed point is unique.

REMARKS. (1) The same argument gives a uniqueness condition for the fixed point theorems of Altman and Rothe [5, Chapter 3]. (2) We thank Dr. John Osborn for help in proving the Lemma.

3. To illustrate our result, we consider the nonlinear boundary value problem

$$\begin{cases} -DC'' + (vC)' = f(x), & 0 < x < 1, \\ C'(0) = 0, & C(1) = a > 0, \end{cases} \tag{1}$$

$$v' = -J(x, C), \qquad v(0) = 0. \tag{2}$$

This system of equations was used by Diamond and Bossert [1] to model salt and water transport in a closed-ended tube, such as a sweat gland. The diffusion coefficient $D > 0$, the function $J(x, C)$ represents an osmotic transport of water out of the tube, and the function $f(x)$ represents a source of salt into the tube. We assume that these functions are sufficiently differentiable, and that

$$f(x) \geqslant 0, \qquad J_C(x, C) < 0, \tag{3}$$

where the subscript C denotes the partial derivative.

We write (1), (2) as a fixed point problem as follows. Let $C_1(x)$ be a continuous function, and with $C = C_1$, let $v(x)$ be the solution of (2). Then with $v(x)$ given, there is a unique solution $C = C_2$ of (1). To see this, we integrate (1) to obtain

$$-DC_2'(x) + v(x)C_2(x) = f_1(x) = \int_0^x f(t)\,dt.$$

Letting v_1 denote an indefinite integral of v, we may then solve this equation to get

$$\begin{aligned} C_2(x) = {} & a\exp\{-D^{-1}[v_1(1) - v_1(x)]\} \\ & + D^{-1}\int_x^1 f_1(t)\exp\{-D^{-1}[v_1(t) - v_1(x)]\}\,dt. \end{aligned} \tag{4}$$

We have thus defined a map $F(C_1) = C_2$ on the Banach space X of continuous functions on $[0, 1]$.

The problem of solving the system (1), (2) is equivalent to the problem of finding a fixed point of F. From (4) and (3) we see that $C_2(x) > 0$. If $C_1(x) \geqslant 0$, we have from (3), $v(x) \geqslant w(x)$, where $w(x) = -\int_0^x J(t, 0)\,dt$. Hence for $t \geqslant x$, $v_1(t) - v_1(x) \geqslant b(t - x)$, where b is independent of $C_1(x)$. Using this in (4), we find that there is a constant K such that, for any $C_1(x) \geqslant 0$, $C_2(x) < K$. If we let $D \subset X$ denote the convex set defined by the inequalities $0 < C(x) < K$, we have proved that $F(\overline{D}) \subset D$. It is easy to see, using (4), that F is continuous, compact, and in fact continuously Fréchet differentiable. Thus from the Schauder fixed point theorem, there is a fixed point $C = F(C)$, and hence a solution $v(x)$, $C(x)$, of (1), (2). To show that there is a unique fixed point, we must calculate the derivative of F. Setting $C_2 = F(C_1)$, $\tilde{C}_2 = F'(C_1)\tilde{C}_1$, it may be verified that $\tilde{C}_1$, $\tilde{C}_2$ satisfy the linear problem

$$- D\tilde{C}_2'' + (\tilde{v}C_2)' + (v\tilde{C}_2)' = 0, \qquad \tilde{C}_2'(0) = \tilde{C}_2(1) = 0,$$
$$\tilde{v}' + J_C(x, C_1)\tilde{C}_1 = 0, \qquad \tilde{v}(0) = 0.$$

To apply our Theorem, we suppose $C_1 \in D$. We must verify that $1 \notin \sigma(F'(C_1))$; that is, that 1 is not an eigenvalue of $F'(C_1)$. Supposing the contrary, let $\tilde{C}$ be an eigenfunction of $F'(C_1)$ with eigenvalue 1. Then $\tilde{C}_2 = \tilde{C}$, and hence there is a nontrivial solution $\tilde{v}(x)$, $\tilde{C}(x)$, of the problem

$$- D\tilde{C}'' + (\tilde{v}C_2)' + (v\tilde{C})' = g_1(x), \tag{5}$$

$$\tilde{v}' + J_C(x, C_1)\tilde{C} = g_2(x), \tag{6}$$

$$\tilde{C}'(0) = \tilde{C}(1) = \tilde{v}(0) = 0, \tag{7}$$

with $g_1(x) \equiv 0$, $g_2(x) \equiv 0$. It may be verified that (5)–(7) defines a closed, densely defined operator on $(\tilde{v}, \tilde{C}) \in L_2(0, 1) \times L_2(0, 1)$, that the resolvent operator is compact, and that the adjoint operator is given by the solution of the problem

$$- \varphi' - C_2\psi' = h_1(x), \tag{8}$$

$$- D\psi'' - v\psi' + J_C\varphi = h_2(x), \tag{9}$$

$$\psi'(0) = \psi(1) = \varphi(1) = 0. \tag{10}$$

Using the Fredholm alternative and the compactness of the resolvent operator associated with the problem (5)–(7), we conclude that there are functions $\varphi(x)$, $\psi(x)$, not identically zero, which satisfy (8)–(11) with $h_1(x) \equiv 0$, $h_2(x) \equiv 0$. We must have $\psi'(1) \neq 0$, since otherwise, by the uniqueness of the solution of the terminal value problem associated with (8), (9), we would have $\varphi \equiv 0$, $\psi \equiv 0$. Let $\bar{x} < 1$ be the largest zero of ψ', and suppose that $\psi'(x) > 0$ in $(\bar{x}, 1)$. Since $C_2 = F(C_1) \in D$, we have $C_2(x) > 0$ in $[0, 1]$. Hence from (8), $\bar{x}$ is the largest zero of φ', and $\varphi'(x) < 0$ in $(\bar{x}, 1)$. Hence $\psi''(\bar{x}) \geqslant 0$, $\varphi(\bar{x}) > 0$. Using (3), we see that the left side of (9) is < 0 at $x = \bar{x}$. This is a contradiction and proves that 1 is not an eigenvalue of $F'(C_1)$. Hence by our Theorem, there is a unique fixed point of F, and a unique solution $v(x)$, $C(x)$ of (1), (2) with $C(x) \geqslant 0$.

REMARK. In [3] there is given a more detailed study of osmotic flow in a tube.

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