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REMARKS ON SOME FIXED POINT THEOREMS

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ABSTRACT. A compact Hausdorff pseudo-topology is introduced on every closed convex bounded subset of a uniformly convex Banach space and is used to prove a previous theorem of the author.

In [7], we used a transfinite induction method which depends on the structure of the real line to prove the following fixed point theorem for multivalued nonexpansive mappings:

THEOREM 1. *Let K be a closed convex bounded nonempty subset of a uniformly convex Banach space and let T: $K \to \mathcal{C}(K)$ be a nonexpansive mapping, where $\mathcal{C}(K)$ denotes the family of nonempty compact (not necessarily convex) subsets of K, equipped with the Hausdorff metric. Then T has a fixed point, i.e. there exists $x \in K$ such that $x \in Tx$.*

The properties of real numbers we used are the order property and the separability, or more explicitly, that a decreasing nonnegative transfinite sequence indexed by ordinals less than the uncountable ordinal Ω must be eventually constant.

Recently, Caristi and Kirk [1], [2], [5] have proven the following theorem and have given several interesting applications:

THEOREM 2 [1], [2]. *Let X be a complete metric space, and let g: $X \to X$ be a self-map of X. Suppose that there exists a lower semicontinuous nonnegative real-valued mapping ξ such that for all x in X,*

$$d(x, g(x)) \leqslant \xi(x) - \xi(g(x)). \tag{1}$$

Then g has a fixed point.

Chi Song Wong [8] has given a simple proof of the Caristi-Kirk theorem by using the transfinite induction method mentioned above. On the other hand, we define a pseudo-compact-Hausdorff-topology on any closed convex bounded subset of a uniformly convex Banach space and give Theorem 1 a simpler and more conceptual proof. It is our feeling that the existence of such a compact Hausdorff pseudo-topology may well serve to explain the similarity between uniform convexity and compactness in some aspects of geometric fixed point theory.

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Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to Δ-converge to a point $x \in X$, written $x_n \to^{\Delta} x$, if

$$\limsup_i d(x_{n_i}, x) \leqslant \limsup_i d(x_{n_i}, y) \tag{2}$$

for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and every $y \in X$. In the terminology of asymptotic center (Edelstein [3], Lim [6]), this says that x is *an* asymptotic center of every subsequence of $\{x_n\}$. $\{x_n\}$ is said to Δ-converge strongly to x if

$$\lim d(x_n, x) \leqslant \liminf d(x_n, y) \quad \text{for every } y \in X. \tag{3}$$

This is equivalent to saying that all subsequences of $\{x_n\}$ have a common asymptotic center ($= x$) and asymptotic radius ($= \lim d(x_n, x)$). In general, x is not unique. That strong Δ-convergence implies Δ-convergence is obvious.

Δ-convergence has the following properties of which only the first two are satisfied by strong Δ-convergence:

(i) if $x_n = x$ for every n, then $x_n \to^{\Delta} x$;

(ii) if $x_n \to^{\Delta} x$, then every subsequence of $\{x_n\}$ Δ-converges to x;

(iii) if $\{x_n\}$ does not Δ-converge to x, then there exists a subsequence of which every subsequence does not Δ-converge to x.

Clearly these definitions and properties can be formulated for nets. A set X equipped with a convergence class satisfying (i), (ii) and (iii) (or only (i) and (ii)) will be called a pseudo-topological space and the convergence class will be called a pseudo-topology on X. Note that we need only one additional axiom to define topological spaces by convergence classes, see e.g., Kelley [4]. By using nets, concepts in topological spaces can be carried over to quasi-topological spaces. Thus a quasi-topological space is compact if every net in it has a convergent subnet and is Hausdorff if a net can converge to at most one point. In what follows, (strong) Δ-topology will refer to the quasi-topology with the convergence class given by sequences satisfying (2) ((3) respectively). One may replace "sequences" by "nets" in this definition and will obtain the same conclusions of Theorems 3 and 4 below.

A metric space is said to be Δ-complete if for every bounded sequence (or net) $\{x_n\}$ in X there is an $x \in X$ such that

$$\limsup_n d(x_n, x) \leqslant \limsup_n d(x_n, y)$$

for every $y \in X$ i.e. $\{x_n\}$ has an asymptotic center in X.

THEOREM 3. *Every bounded Δ-complete metric space X is strongly Δ-compact (and hence Δ-compact), i.e. every sequence in X has a strongly Δ-convergent subsequence.*

THEOREM 4. *Every closed convex bounded subset of a uniformly convex Banach space is compact Hausdorff under the (strong) Δ-topology.*

Our proofs of Theorems 3 and 4 need the following set-theoretical result.

For two sequences $\{x_n\}$ and $\{y_n\}$ in a set, let us say $\{x_n\}$ is an essential subsequence of $\{y_n\}$ if there exists a positive integer N such that $\{x_n\}_{n \geqslant N}$ is a subsequence of $\{y_n\}$.

PROPOSITION 1. *Let X be a set and let $\{x_n\}$ be a sequence in x. Let r be a real-valued function whose domain is the set of subsequences of $\{x_n\}$. Suppose $r(y) \leqslant r(z)$ whenever y is an essential subsequence of z. Then there is a subsequence w of $\{x_n\}$ such that $r(z) = r(w)$ for every subsequence z of w.*

PROOF. Denote by $\mathfrak{F}$ the family of subsequences of $\{x_n\}$. Define an ordering $\leqslant$ on $\mathfrak{F}$ as follows:

For $x, y \in \mathfrak{F}$, we put $x < y$ if x is an essential subsequence of y and $r(x) < r(y)$. Then we say $x \leqslant y$ if $x < y$ or x is identically equal to y.

It is easy to check that $\leqslant$ is a reflexive, antisymmetric and transitive relation. Let $\mathcal{C}$ be a chain in $\mathfrak{F}$. Let $r = \inf\{r(x)\colon x \in \mathcal{C}\}$. If there is an $x \in \mathcal{C}$ such that $r(x) = r$, then x is a lower bound for $\mathcal{C}$. Therefore we assume that such x does not exist. Let x_n be a sequence in $\mathcal{C}$ such that $r(x_n)$ strictly decreases to r. Since $\mathcal{C}$ is a chain, we must have $x_1 > x_2 > \dots$. By using the diagonal process, dropping a finite number of terms in each sequence X_n if necessary, we obtain a sequence y which is an essential subsequence of x_n for every n. Then, by assumption, $r(y) < r(x_n)$ for every n and, hence, $r(y) < r(x)$ for every $x \in \mathcal{C}$. Since each $x \in \mathcal{C}$ is an essential subsequence of x_n for some n, we conclude that y is a lower bound for $\mathcal{C}$.

By Zorn's lemma, $\mathfrak{F}$ has a minimal element z. Let w be a subsequence of z. Then $r(w) \leqslant r(z)$. If $r(w) < r(z)$, then $w \leqslant z$ and by the minimality and antisymmetry we must have $w - z$ which implies $r(w) = r(z)$, a contradiction. Hence $r(w) = r(z)$. Q.E.D.

PROOF OF THEOREM 3. Let $\{x_n\}$ be a sequence in X. For every subsequence $\{x_{n_i}\}$, let

$$r(\{x_{n_i}\}) = \inf\left\{ \limsup_i d(x_{n_i}, y) \colon y \in X \right\}.$$

By Proposition 1 $\{x_n\}$ contains a subsequence which we still denote by $\{x_n\}$ such that

$$r(\{x_n\}) = r(\{x_{n_i}\}) = r$$

for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$. By Δ-completeness, there exists an $x \in X$ such that

$$\limsup d(x_n, x) = r(\{x_n\}).$$

For every subsequence $\{x_{n_i}\}$ of $\{x_n\}$, we have

$$\begin{aligned} \limsup d(x_{n_i}, x) &\leqslant \limsup d(x_n, x) \\ &= r(\{x_n\}) = r(\{x_{n_i}\}) \leqslant \limsup d(x_{n_i}, x); \end{aligned}$$

thus

$$\limsup d(x_{n_i}, x) = r(\{x_{n_i}\}).$$

This shows that all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ have a same asymptotic center x and a same asymptotic radius r. Q.E.D.

PROOF OF THEOREM 4. This follows from Theorem 3 and the uniqueness of asymptotic center as proved by Edelstein [3]. Q.E.D.

Let us now give

PROOF OF THEOREM 1. By a standard argument, there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in Tx_n$ and $\|x_n - y_n\| \to 0$. By Theorem 4, $\{x_n\}$ has a Δ-convergent subsequence which we still denote by $\{x_n\}$. Let x be its Δ-limit. We assert that $x \in Tx$. For each n, choose $p_n \in Tx$ such $\|p_n - y_n\| \leqslant \|x - x_n\|$. Since Tx is compact, there exists a convergent subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \to p$ for some $p \in Tx$. It can be easily shown, by using $\|x_n - y_n\| \to 0$, $\|p_n - y_n\| \leqslant \|x - x_n\|$ and $x_n \to^{\Delta} x$, that $x_{n_i} \to^{\Delta} p$. Since also $x_{n_i} \to^{\Delta} x$, we must have $x = p \in Tx$ by the uniqueness of Δ-limit. Q.E.D.

REMARK. I am indebted to the referee for informing me that K. Goebel has discovered independently a similar proof of Theorem 1 in a paper published in Ann. Univ. Mariae Curie-Skłodowska.

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