THE UNDECIDABILITY OF A FUNDAMENTAL PROBLEM IN CLUSTER SET THEORY

J. A. EIDSWICK

Abstract. The undecidability of the existence of a nonmetrizable normal picket fence space is established and used to establish the undecidability of the following statement: Any family of approach curves (approaching a point in $\mathbb{R}^2$) along which cluster sets can be arbitrarily preassigned has a nonintersecting truncation.

1. Let $\Gamma$ denote the set of all curves in $\mathbb{R}^2$ that approach the origin, and for $\gamma \in \Gamma$ and $f: \mathbb{R}^2 \to \mathbb{R}$, let $C(f, \gamma)$ denote the cluster set of $f$ at 0 along the curve $\gamma$ (i.e., $C(f, \gamma) = \{ \lambda : \text{there exists a sequence } \{P_n\} \text{ of distinct points on } \gamma \text{ such that } P_n \to 0 \text{ and } f(P_n) \to \lambda \}$). A search in [2] for large subsets $\Gamma_0$ of $\Gamma$ along which cluster sets could be arbitrarily preassigned led to the following conjecture:

(T) A subfamily $\Gamma_0$ of $\Gamma$ has the property that for any given collection $\{C(\gamma) : \gamma \in \Gamma_0\}$ of closed sets there exists a function $f$ such that $C(f, \gamma) = C(\gamma)$ for every $\gamma$ in $\Gamma_0$ if and only if $\Gamma_0$ has a nonintersecting truncation.

In [3] statement (T) was shown to be equivalent to the restriction of the normal Moore space conjecture to a special class of spaces. Consequently (see [3]), (T) is consistent with the usual axioms of set theory (ZFC). The purpose of this paper is to show that the negation of (T) is also consistent. This will be done by establishing the existence of a nonmetrizable normal picket fence space. Such a space will then be transformed into a counterexample to (T). Martin’s Axiom plus the negation of the continuum hypothesis (MA + ¬ CH) will be assumed (cf. [8]).

2. A nonmetrizable normal picket fence space. As in [3], for $S \subset \mathbb{R}$ and $m: S \to \mathbb{R}$, let $(S, m)$ denote the family of half-lines $y = m(s)x + s, x > 0, s \in S$, and let $X(S, m)$ denote the associated picket fence space (the points of $X(S, m)$ are the points of the right half-plane $x > 0$ together with the points of $S$; the right half-plane has the discrete topology and basic neighborhoods of points $s$ of $S$ are the truncated lines $y = m(s)x + s, 0 \leq x < \tau, \tau > 0$).

The main difference between these spaces and the familiar tangent disk
A FUNDAMENTAL PROBLEM IN CLUSTER SET THEORY

spaces (see [9]) is that these spaces generally do not have the countable
cellularity property on the discrete subspace S (for an extreme case, see [4]).
Hence, the usual method of obtaining a normal noncollectionwise Hausdorff
space from a Q-set does not apply to picket fence spaces. Also, it is not at all
clear how a nonmetrizable normal space formed from triangular disks or
wedges could be converted into one formed from pickets. Thus, it appears that
a special approach like the one given below is necessary.

**THEOREM (MA + \neg CH).** There exists a nonmetrizable normal picket fence space.

**Proof.** Let S be any set of real numbers of power \(\aleph_1\). By [7, pp. 38–40] and
[5, p. 154], MA + \neg CH implies that S is a Q-set and has Lebesgue measure
zero. Note that \(X(S, m)\) is normal no matter how \(m\) is defined (cf. [1, Example
E]).

Let \(\phi\) be a function from \(\mathbb{R}\) to \(\mathbb{R}\) whose derivative is equal to \(-\infty\) at every
point of S. This is possible because S has measure zero (see [6, p. 214]). Let \(m\)
be the restriction of \(\phi\) to S. Observe that if \(X(S, m)\) were collectionwise
Hausdorff, then for some uncountable subset \(A\) of S and some positive
number \(\epsilon\), the family \((A, m)\) would be nonintersecting on the strip \(0 < x < \epsilon\).
The following argument shows that this is not the case.

Let \(A\) be an uncountable subset of S and let \(a\) be a point of \(A\) which is also
a limit point of \(A\). Since \(\phi'(a) = -\infty\), there exists a sequence \(\{c_n\}\) in \(A\) such
that \((\phi(c_n) - \phi(a))/(c_n - a) \leq -n\) for every \(n\). Hence, there exist sequences
\(\{a_n\}\) and \(\{b_n\}\) in \(A\) such that \(0 < (b_n - a_n)/(m(a_n) - m(b_n)) < 1/n\) for every
\(n\). In other words, the family \((A, m)\) intersects itself on every strip of the form
\(0 < x < 1/n\).

Since metrizability and collectionwise normality are equivalent for Moore
spaces [1], the theorem follows.

3. A counterexample to (T). In [3] it was shown that statement (T) is
equivalent to the statement: Every normal Moore space \(X(\Gamma_0)\) is metrizable.
(The points of \(X(\Gamma_0)\) are the points of the punctured plane \(\mathbb{R}^2 - \{0\}\) together
with the curves of \(\Gamma_0\); the punctured plane has the discrete topology and basic
neighborhoods of a point \(\gamma\) in \(\Gamma_0\) are the truncations of \(\gamma\).)

**THEOREM (MA + \neg CH).** There exists a nonmetrizable normal Moore space
\(X(\Gamma_0)\).

**Proof.** Let \(X(S, m)\) be a nonmetrizable normal picket fence space and let
\(\Gamma_0\) be the family of parabolic approaches \(y = m(s)x^2 + sx, x > 0, s \in S\).
Obviously, \(X(\Gamma_0)\) is a Moore space. Normality and noncollectionwise normal-
ity follow from the fact that the truncated lines \(y = m(s)x + s, 0 < x < \sigma,\)
and \(y = m(t)x + t, 0 < x < \tau,\) intersect if and only if the truncated parabo-
las \(y = m(s)x^2 + sx, 0 < x < \sigma,\) and \(y = m(t)x^2 + tx, 0 < x < \tau,\) intersect.

**Remark.** In the above theorems, the full strength of MA + \neg CH is not
needed; all that is needed is the existence of a \( Q \)-set (cf., e.g., Theorem 5(b) of [7]).

REFERENCES