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ON THE SPACE OF FUNCTIONS WITHOUT DISCONTINUITIES OF THE SECOND KIND

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ABSTRACT. In this note we prove a general theorem which implies the famous proposition that the space of functions without discontinuities of the second kind, equipped with the Skorohod metric, is homeomorphic to a complete metric space.

1. Let $D[0, 1]$ be the set of all functions $x(t)$, $0 \leqslant t \leqslant 1$, without discontinuities of the second kind. We assume that the function $x(t) \in D[0, 1]$ is continuous from the right at all points $0 \leqslant t < 1$ and $x(t)$ is continuous from the left at 1. Denote by Λ the set of all continuous and strictly increasing functions $\lambda(t)$, $0 \leqslant t \leqslant 1$, such that $\lambda(0) = 0$, $\lambda(1) = 1$. We shall consider $D[0, 1]$ with the Skorohod metric, namely,

$$\rho_s(x_1, x_2) = \inf_{\lambda \in \Lambda} \left[\sup_t |x_1(\lambda(t)) - x_2(t)| + \sup_t |\lambda(t) - t| \right].$$

The space $D[0, 1]$ is separable, but it is not a complete space. A space is said to be topologically complete if it is homeomorphic to a complete metric space. Several proofs of the topological completeness of $D[0, 1]$ have been given (see, for example, [3, 3.14]). These proofs utilize the existence for $D[0, 1]$ of a family of functionals $\Delta_c(x)$ ($x \in D[0, 1]$, $c > 0$) which can be used to prove an analog to the Arzela-Ascoli Theorem to characterize the compact sets in $D[0, 1]$. We shall prove that for an arbitrary separable metric space the existence of such an "Arzela-Ascoli type" family of functions is both necessary and sufficient to insure topological completeness.

2. THEOREM. *The separable metric space Z is topologically complete if and only if there exists a family $G_c(z)$ ($c > 0$) of bounded continuous functions defined on Z such that*:

(1) $G_c(z) \geqslant 0$;

(2) *for a fixed z we have* $\lim_{c \to 0} G_c(z) = 0$;

(3) $G_{c_1}(z) \leqslant G_{c_2}(z)$ *if* $c_1 \leqslant c_2$;

(4) *the closed set $K \subset Z$ is compact if and only if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* $z \in K$ *and each* $c < \delta$ *we have* $G_c(z) < \varepsilon$.

PROOF. *Necessity*. The space Z is homeomorphic to the separable complete metric space Z'. Denote by $C[0, 1]$ the space of continuous functions $y(t)$,

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defined on [0, 1], with the usual metric $\rho(y_1, y_2) = \max|y_1(t) - y_2(t)|$. The Banach-Mazur Theorem asserts that a separable metric space is isometric to a subset of a space $C[0, 1]$ (see [2, §65]). Let Z' be isometric to $Z'' \subset C[0, 1]$. Since Z' is a complete space, it follows that Z'' is a closed subset of $C[0, 1]$. Consider for each $c > 0$ the following functional on Z'':

$$G_c(y) = \min\Big\{ \sup_{|t'-t''|<c} |y(t') - y(t'')|, 1 \Big\} + \min\{ c \cdot \max|y(t)|, 1 \}.$$

The functionals $G_c(y)$ may be considered as functions on Z: $G_c(z)$. Obviously conditions (1)–(3) are satisfied for $G_c(z)$. Condition (4) is valid, according to the Arzela-Ascoli Theorem.

Sufficiency. Let $G_c(z)$ be a family of bounded continuous functions defined on the separable metric space Z, which satisfies conditions (1)–(4). By virtue of Urysohn's Theorem (see [2, §58]) the space Z is homeomorphic to a certain subset of Hilbert space H. For every element $z \in Z$ let $\Phi(z)$ denote the following element of H: $(f_1(z), 2^{-1}f_2(z), \ldots, 2^{-n+1}f_n(z), \ldots)$, where f_n is defined in [2, p. 128]. The set $\Phi(Z)$ is homeomorphic to Z. Let $G_c'(z) = G_c(z)/(A_c + 1)$, where $A_c = \sup_z G_c(z)$.

For every element $z \in Z$ let $\Phi'(z)$ denote the following element of H:

$$\big(f_1(z), G_1'(z), 2^{-1}f_2(z), 2^{-1}G_{1/2}'(z), \ldots, 2^{-n+1}f_n(z), 2^{-n+1}G_{1/n}'(z), \ldots\big).$$

The set $\Phi'(Z)$ is homeomorphic to Z. The set $Q = \overline{\Phi'(Z)}$ is the metric compactification of Z such that all functions $G_{1/n}(z)$ (n a positive integer) can be continuously extended on Q. Consider the closed subsets in Q: $F_{m,n} = \{q \in Q: G_{1/n}(q) \geqslant 1/m\}$. Set $F_m = \bigcap_{n=1}^{\infty} F_{m,n}$ and $F_\sigma = \bigcup_{m=1}^{\infty} F_m$. Then $F_\sigma = Q \setminus Z$. Indeed, it is obvious that $F_\sigma \subset Q \setminus Z$. Suppose that there exists a point $q_0 \in (Q \setminus Z) \setminus F_\sigma$. Consider the sequence of points $Z_\infty = \{z_p\} \subset Z$, which converges to q_0 in the metric of Q. The set Z_∞ is closed in Z. For any m there exists n_1 such that $G_{1/n_1}(q_0) < 1/m$. Consider the open set in Q: $U = \{q \in Q: G_{1/n_1}(q) < 1/m\}$. There exists a positive integer P such that for $p \geqslant P$ we have $z_p \in U$. There exists also a positive integer $n_2 \geqslant n_1$ such that $G_{1/n_2}(z_p) < 1/m$ for $p < P$. Hence, $G_{1/n_2}(z_p) < 1/m$ for $z_p \in Z_\infty$. This means that the sequence Z_∞ is compact, which is a contradiction. Thus $F_\sigma = Q \setminus Z$. From Alexandroff's Theorem (see [1, 11.2]) it follows that Z is topologically complete. Q.E.D.

3. Consider the space $D[0, 1]$. The functionals $\Delta_c(x)$ defined in [4, VI, §5] can be altered to yield a family of continuous bounded functionals $g_c(x)$ satisfying the conditions of the Theorem by setting

$$g_c(x) = \min\big[F_{1/c}(x), 1\big] + \min\big[c \cdot \sup|x(t)|, 1\big],$$

where F_a is as defined in [4, p. 430]. This means that $D[0, 1]$ is topologically complete.

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