COMPRESSIBLE MAPS

JAY E. GOLDFEATHER

Abstract. Weingram has shown that if \( G \) is a finitely generated abelian group, then every nontrivial map \( f: \Omega S^{2n+1} \to K(G,2n) \) is incompressible; that is, \( f \) is not homotopic to a map whose image is contained in some finite-dimensional skeleton.

It is shown that a nontrivial map \( \Omega S^{2n+1} \to K(G,2n) \) may be compressible if \( G \) is not finitely generated. This result leads to some understanding of the obstructions to compressibility in Weingram's Theorem.

A map \( f: X \to Y \) is said to compress into \( A \subseteq Y \) if there is a map \( f': X \to A \) such that \( f \simeq i \cdot f' \), where \( i \) is inclusion. Weingram [4] has shown that if \( G \) is a finitely generated abelian group, then every nontrivial map \( f: \Omega S^{2n+1} \to K(G,2n) \) is incompressible, that is, \( f \) is not homotopic to a map whose image is contained in some finite-dimensional skeleton.

It will be shown that a nontrivial map \( \Omega S^{2n+1} \to K(G,2n) \) may be compressible if \( G \) is not finitely generated. Specifically, if \( P \) is a set of primes and \( Z_p \subseteq Q \) is the subgroup of \( P \)-local integers, then any map \( \Omega S^{2n+1} \to K(Q/Z_p,2n) \) compresses into the \((2n+1)\)-skeleton, provided \( 2 \notin P \) or \( n = 1, 3 \).

Let \( M(G,k) \) denote a Moore space of type \((G,k)\). For the remainder of this paper, it will be assumed that \( 2 \notin P \) or \( 2 \in P \) and \( n = 1, 3 \).

The following two well-known theorems will be stated without proof:

**Theorem (Adams [1]).** \( M(Z_p,2n+1) \) is an H-space.

**Theorem (Stasheff [3]).** Let \( X \) and \( W \) be H-spaces and let \( F: X \to W \) be an H-map. Let \( \Omega W \to Y \to X \) be the fibration induced by \( f \). Then \( Y \) is an H-space.

**Lemma 1.** Any map \( M(Z_p,2n+1) \to K(Q,2n+1) \) is homotopic to an H-map.

**Proof.** The obstructions to a map \( f: X \to Y \) being homotopic to an H-map lie in \( H^i(X \wedge X; \pi(Y)) \). Since

\[ \pi_i(K(Q,2n+1)) = 0 \quad \text{for} \ i \neq 2n+1 \]

and

\[ H^{2n+1}(M(Z_p,2n+1) \wedge M(Z_p,2n+1); G) = 0 \quad \text{for any} \ G, \]

all obstructions lie in zero groups.
In view of the preceding lemma, it will be assumed that any given map
\( f: M(Z_p, 2n + 1) \to K(Q, 2n + 1) \) is an \( H \)-map.

Let \( f \) be induced by the natural embedding \( Z_p \subseteq Q \).

**Proposition 2.** Let \( K(Q, 2n) \to E \to M(Z_p, 2n + 1) \) be the fibration induced
by \( f \). Then \( E \) is an \( H \)-space.

**Proof.** This is an immediate consequence of the above two theorems and
Lemma 1.

**Theorem 3.**

\[
\tilde{H}_i(E; Z) = \begin{cases} 
Q/Z_p, & i = 2n, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** (Porter [2] has proved a similar theorem when the base space is \( S^3 \))
\( H_\bullet(K(Q, 2n); Z) \) is a polynomial algebra over \( Z \) on one generator in dimension \( 2n \) tensored with \( Q \). Since \( f \) is induced by the natural embedding of \( Z_p \subseteq Q \), the Serre exact sequence

\[
H_{2n+1}(E) \to H_{2n+1}(M(Z_p, 2n+1)) \to H_{2n}(K(Q, 2n)) \to H_{2n}(E) \to 0
\]

reduces to

\[
0 \to Z_p \to Q \to H_{2n}(E) \to 0
\]

and, hence, \( H_{2n}(E) = Q/Z_p \).

In the homology spectral sequence, the only nonzero differential is \( d^{2n+1} \)
which is an isomorphism for total degrees greater than \( 2n + 1 \) since \( Z_p \otimes Q \)
\( \cong Q \). Hence, \( H_i(E) = 0 \) for \( i > 2n + 1 \).

**Remark.** \( E \) is homotopic to a \((2n+1)\)-dimensional Moore space

\( M(Q/Z_p, 2n) \).

**Proposition 4.** \( K(Q/Z_p, 2n) \) is not homotopic to a finite-dimensional complex.

**Proof.** The exact sequence \( 0 \to Z_p \to Q \to Q/Z_p \to 0 \) induces a fibration

\[ K(Q/Z_p, 2n) \to K(Z_p, 2n + 1) \to K(Q, 2n + 1). \]

It is well known that \( K(Q, 2n + 1) \) is homotopic to a finite-dimensional complex. Hence if \( K(Q/Z_p, 2n) \) were also homotopic to a finite-dimensional complex, it would imply that \( H^k(K(Z_p, 2n + 1); Z_p) = 0 \) for all \( k \) greater than some integer. It suffices to show, then, that \( H^k(K(Z_p, 2n + 1); Z_p) \) is nonzero for infinitely many \( k \), where \( p \in P \).

Let \( \iota \) be the fundamental class in \( H^{2n+1}(K(Z, 2n + 1); Z_p) = \text{Hom}(Z, Z_p) \).
It is well known that there are an infinite number of Steenrod operations \( \Theta_I \)
such that \( \Theta^I \iota \neq 0 \). (\( I \) denotes an admissible sequence.) Hence it suffices to show that if \( f: K(Z, 2n + 1) \to K(Z_p, 2n + 1) \) is the natural inclusion, then
THEOREM 5. Every nontrivial map \( \Omega S^{2n+1} \to K(Q/ZP, 2n) \) compresses into the \((2n+1)\)-skeleton.

PROOF. Any map \( f: \Omega S^{2n+1} \to K(Q/ZP, 2n) \) is homotopic to a map \( r \cdot \Omega Sg \), where \( g: S^{2n} \to K(Q/ZP, 2n) \) and \( r \) is the retraction \( \Omega SK(Q/ZP, 2n) \to K(Q/ZP, 2n) \).

But every such map \( g \) factors through \( M(Q/ZP, 2n) \) so that

\[
S^{2n} \xrightarrow{g_1} M(Q/ZP, 2n) \xrightarrow{g_2} K(Q/ZP, 2n)
\]

is nontrivial where

\[
g \simeq g_2 \cdot g_1.
\]

By Theorem 3 and Proposition 2, \( M(Q/ZP, 2n) \) is an \( H \)-space so that \( g_1 \) extends to \( \bar{g}_1: \Omega S^{2n+1} \to M(Q/ZP, 2n) \). Then \( f \simeq g_2 \cdot \bar{g}_1 \), so \( f \) compresses into the \((2n+1)\)-skeleton.

Observe that if \( \mathbb{P} = \{p\} \), then \( Q/ZP = Z_{px} = \lim_r Z_{p^r} \) induced by the inclusion \( Z_{p^r} \to Z_{p^{r+1}} \). Let \( G \) be a finitely-generated odd torsion group so that \( G = \bigoplus_{i=1}^m Z_{p_i^n} \). Let \( P = \{p_1, \ldots, p_m\} \) and \( G_{k+1} = \bigoplus_{i=1}^m Z_{p_i^{n+k}} \). Then \( Q/ZP \cong \lim_k G_k \).

Since \( \pi_{2n+1}(M(G_r, 2n)) = G_r \otimes Z_2 = 0 \), and \( M(G_r, 2n) \) can be thought of as the \((2n+1)\)-skeleton of \( K(G_r, 2n) \), the obstructions to extending

\[
S^{2n} \to K(G_r, 2n)^{(2n+1)}
\]

to \( \Omega S^{2n+1} \to K(G_r, 2n)^{(2n+1)} \) are the obstructions to extending

\[
S^{2n} \to M(G_r, 2n)
\]
to \( \Omega S^{2n+1} \to M(G_r, 2n) \).

Let \( j_r, m: M(G_r, 2n) \to M(G_{r+m}, 2n) \) be inclusion and let \( j: S^{2n} \to M(G_r, 2n) \).

THEOREM 6. For every \( k \) and \( r \), there is an \( m \) such that \( j_r, m \cdot j \) extends to \( (S^{2n})_k \), the \( k \)th reduced product of \( S^{2n} \).

PROOF. Let \( j_{r, \infty}: M(G_r, 2n) \to M(Q/ZP, 2n) \) be the inclusion. Since \( M(Q/ZP, 2n) \) is an \( H \)-space, it follows that \( j_{r, \infty} \cdot j \) extends to \( (S^{2n})_k \) for all \( k \).

Let \( j \) be such an extension. Since \( (S^{2n})_k \) is compact, image \( j \subseteq M(G_{r+m}, 2n) \) for some \( m \) and, hence, \( j \) is an extension of \( j_{r, m} \cdot j \).


Department of Mathematics, University of Wisconsin, Milwaukee, Wisconsin 53201