SOME NOWHERE EQUICONTINUOUS
HOMEOMORPHISMS\textsuperscript{1}

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Abstract. It is shown that a nowhere equicontinuous homeomorphism can
be defined on a compact polyhedron \( X \) if and only if \( X \) does not have cell
decomposition which contains a principal 1-cell. It is also shown that for
each locally connected contractible continuum \( C \) in the plane, there is a
nowhere equicontinuous homeomorphism \( h_c \) on a disk in the plane such that
the fixed point set of \( h_c \) is \( C \).

1. Introduction. Let \( X \) be a metric space with a metric \( d \) and \( h \) a
homeomorphism on \( X \) (a homeomorphism of \( X \) onto itself). We say that \( h \) is
equicontinuous at \( x \in X \) if \( \{h^n|n \in \mathbb{Z}\} \) is an equicontinuous family at \( x \). The
set \( \{x \in X|h \text{ is equicontinuous at } x\} \) is called the regular set of \( h \) and its
complement in \( X \) is called the irregular set of \( h \). If the regular set of \( h \) is empty,
we say that \( h \) is a nowhere equicontinuous homeomorphism (NEH). Homeomor-
phisms \( h_1 \) and \( h_2 \) on \( X \) are said to be topologically equivalent if there is a
homeomorphism \( k \) on \( X \) such that \( h_1 = k^{-1}h_2k \).

It is a known fact that neither the closed unit interval nor the circle admit
a NEH \([2]\). In fact, it is true that if \( h: X \to X \) is a homeomorphism on \( X \),
where \( X \) is either the closed unit interval or the circle, then the irregular set of
\( h \) is nowhere dense in \( X \). It can be easily shown that, for compact spaces, the
property of admitting a NEH is topological. In \( \S 2 \) of this paper we show that
a compact polyhedron admits a NEH if and only if it does not have a cell
decomposition which contains a principal 1-cell (Theorem 5). We also show
that for each locally connected contractible continuum \( C \) in the interior of the
unit disk, with \( \text{diam} \ (C) > 0 \), there is a NEH \( h^* \) on the unit disk such that
\( \text{Fix} (h^*) = C \) where \( \text{Fix} (h^*) = \{x|h^*(x) = x\} \) (Theorem 7).

Throughout this paper we use such standard terminologies as orbit, dense
orbit, periodic point and refer readers to \([1]\) for definitions. We also use some
standard terminologies of piecewise linear topology and refer readers to \([4]\)
for their definitions. The symbols \( I, B^n, S^n, \partial X \) and \( \partial X \) are used to denote the
closed unit interval \([0, 1]\), the \( n \)-ball, the \( n \)-sphere, combinatorial interior of \( X \)
and the combinatorial boundary of \( X \) respectively. A principal \( n \)-cell in a complex is an \( n \)-cell which intersects higher dimensional cells in a subset of its boundary. All spaces considered here are metric spaces and a map is a continuous function.

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2. The existence of NEH's. It can be easily shown that the homeomorphism \( h \) on \( S^1 \), the unit circle in the complex plane, defined by \( h(e^{i\theta}) = e^{i(\theta + 2\pi t)} \), \( 0 \leq t \leq 1 \), is periodic with the period \( q \) if \( t \) is rational and \( t = P/q \) in the lowest term, and each point of \( S^1 \) has dense orbit in \( S^1 \) under \( h \) if \( t \) is irrational. Now we proceed with the construction of a NEH on some compact polyhedron.

Lemma 1. Define \( h: S^1 \times I \to S^1 \times I \) by \( h(e^{i\theta}, t) = (e^{i(\theta + 2\pi t)}, t) \). Then \( h \) is a NEH on \( S^1 \times I \).

Proof. Assume that the metric \( d \) on \( S^1 \times I \) is the product metric. Let \( x \in S^1 \times I \) and write \( x = (e^{i\theta}, t), 0 \leq \theta \leq 2\pi, t \in I \). Take \( \varepsilon = \sqrt{2} \). For each \( \delta \), take \( t' \) such that \( 0 < |t - t'| < \min\{\delta, \frac{1}{4}\} \). Then \( 0 < d((e^{i\theta}, t), (e^{i\theta}, t')) < \delta \) and there is an integer \( n \) such that \( \frac{1}{4} \leq n(t - t') \leq \frac{1}{2} \). Thus

\[
d(h^n(e^{i\theta}, t), h^n(e^{i\theta}, t')) = d((e^{i(\theta + 2\pi nt)}, t), (e^{i(\theta + 2\pi nt'}), t'))^2 \\
\geq d((e^{i(\theta + 2\pi nt)}, t), (e^{i(\theta + 2\pi nt')}, t'))^2 = 2 \sin n\pi(t - t') \geq \sqrt{2}
\]

by the choice of \( n \).

Corollary 2. Let \( X \) be a space. If there is a map \( f: X \to I \) such that \( \text{Int} (f^{-1}(t)) \), the point set interior of \( f^{-1}(t) \), is empty, for each \( t \in I \), then \( S^1 \times X \) admits a NEH.

Proof. Assume that \( S^1 \times X \) has the product metric. Define \( g: S^1 \times X \to S^1 \times X \) by \( g((e^{i\theta}, x)) = (e^{i(\theta + \pi f(x))}, x) \). Take \( \varepsilon = \sqrt{2} \). For each \( (e^{i\theta}, x) \in S^1 \times X \) and any neighborhood \( U \) of \( (e^{i\theta}, x) \), there is a \( \delta > 0 \) such that the \( \delta \)-neighborhood \( N_\delta(x) \) of \( x \) is contained in \( \pi_x(U) \), where \( \pi_x \) is the projection map of \( S^1 \times X \) onto \( X \). Thus, there is a point \( y \neq x \) in \( N_\delta(x) \) such that \( f(x) \neq f(y) \), since \( \text{Int} (f^{-1}(t)) = \emptyset \) for each \( t \). Therefore \( 0 < |f(x) - f(y)| < 1 \) so that \( \frac{1}{2} < n(f(x) - f(y)) \leq 1 \) for some integer \( n \). Then

\[
d(h^n(e^{i\theta}, x), h^n(e^{i\theta}, y)) = d((e^{i(\theta + \pi nf(x))}, x), (e^{i(\theta + \pi nf(y))}, y))^2 \\
\geq d((e^{i(\theta + \pi nf(x))}, x), (e^{i(\theta + \pi nf(y))}, x)) \geq \sqrt{2}.
\]

Lemma 3. There is a NEH \( \xi_2 \) on \( B^2 \) such that \( \xi_2 \upharpoonright \partial B^2 \), the restriction of \( \xi_2 \) to \( \partial B^2 \), is \( 1_{\partial B^2} \), the identity map of \( \partial B^2 \).

Proof. Let \( f: S^1 \times I \to B^2 \) be a map which satisfies the following condi-
tions: \( f|_{S^1 \times [0,1]} \) is a homeomorphism of \( S^1 \times (0,1) \) onto \( B^2 - \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\} \), \( f((e^\theta,0)) = f((e^{(2\pi - \theta)})) \), \( 0 \leq \theta \leq 2\pi \), \( f((e^{\pi},0)) = (-\frac{1}{2},0) \), \( f((e^{0},0)) = (\frac{1}{2},0) \) and \( f|_{\{(e^\theta,0)|0 \leq \theta \leq \pi\}} \) is a homeomorphism of \( \{(e^\theta,0)|0 \leq \theta \leq \pi\} \) onto \( \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\} \). Take \( h: S^1 \times I \rightarrow S^1 \times I \) defined in Lemma 1. Define \( \xi_2: B^2 \rightarrow B^2 \) by \( \xi_2(x) = fh^{-1}(x) \). Then it is easy to see that \( \xi_2 \) is a homeomorphism on \( B^2 \). If \( p \in B^2 - \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\} \) then \( p = f((e^\theta,0)) \) for some \( t \neq 0 \). Since \( f|_{S^1 \times [t/2,1]} \) is a homeomorphism of \( S^1 \times [t/2,1] \) onto an annulus \( A \subset B^2 - \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\} \), both \( f|_{S^1 \times [t/2,1]} \) and \( f^{-1}|_A \) are uniformly continuous. Thus, if \( \xi_2|_A \) were equicontinuous at \( p \) then \( h|_{S^1 \times [t/2,1]} \) would be equicontinuous at \( (e^\theta,t) \). Therefore \( \xi_2|_A \) is not equicontinuous at \( p \) so that \( \xi_2 \) is not equicontinuous at \( p \). If \( p \in \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\} \), then \( \xi_2(p) = p \). Choose \( \varepsilon > 0 \) so that \( N_{2\varepsilon}(p) \) does not contain \( \{(x,0)|-\frac{1}{2} \leq x \leq \frac{1}{2}\} \), then \( \xi_2(p) = p \). Then there is an \( \eta \) and a neighborhood \( U \) of \( e^n \) such that \( \pi \times [0,1] \subset S^1 \times [0,1] \). Then \( f^{-1}(N_{\eta}(p)) \) which has dense orbit in \( S^1 \times [t] \) under \( h \). Then there is an integer \( n \) such that \( h^n((e^\theta,t)) \in U \times \{t\} \). Therefore \( \xi_2(f((e^\theta,t))) \notin N_{\eta}(p) \) and \( f((e^\theta,0)) \in N_{\eta}(p) \) which shows that \( \xi_2 \) is not equicontinuous at \( p \). It is clear that \( \xi_2|_{B^2} = i_3|_{B^2} \).

**Lemma 4.** For each \( n \geq 2 \), \( B^n \) admits a NEH \( \xi_n \) such that \( \xi_n|_{B^n} = i_3|_{B^n} \).

**Proof.** We prove this lemma by induction on \( n \). By Lemma 3, \( B^2 \) admits such a homeomorphism. Assume that there is such a homeomorphism \( \xi_{n-1} \) on \( B^{n-1} \). For each \( \theta, 0 \leq \theta < 2\pi \), let

\[
B^{n-1}_\theta = \left\{(x_1, \ldots, x_{n-2}, x_{n-1} \cos \theta, x_{n-1} \sin \theta) \mid \sum_{i=1}^{n-1} x_i^2 \leq 1 \text{ and } x_{n-1} > 0 \right\}.
\]

Then \( B^{n-1}_\theta \) is the closed half of the unit ball sitting in the subvector space in \( R^n \) of dimension \( n-1 \) which is determined by \( R^{n-2} \) and the vector \( (0, \ldots, 0, \cos \theta, \sin \theta) \in R^n \). Thus it is easy to see that

\[
B^n = \bigcup_{0 \leq \theta < 2\pi} B^{n-1}_\theta \text{ and } B^{n-1}_\theta \cap B^{n-1}_{\theta'} = B^{n-2}
\]

for \( \theta \neq \theta' \). Since \( (B^{n-1}_\theta, \partial B^{n-1}) \) and \( (B^{n-1}_{\theta'}, \partial B^{n-1}_{\theta'}) \) are homeomorphic as compact pairs, there is a NEH \( \psi_{n-1}: B^{n-1}_{\theta} \rightarrow B^{n-1}_{\theta'} \) such that \( \psi_{n-1}|_{\partial B^{n-1}_\theta} = i_{3B^{n-1}_\theta} \). Define \( \xi_n: B^n \rightarrow B^n \) by \( \xi_n|_{B^{n-1}_\theta} = \rho_\theta \psi_{n-1} \rho_\theta^{-1} \) where \( \rho_\theta: B^{n-1}_0 \rightarrow B^{n-1}_\theta \) is the homeomorphism defined by

\[
\rho_\theta((x_1, \ldots, x_{n-2}, x_{n-1}, 0)) = (x_1, \ldots, x_{n-2}, x_{n-1} \cos \theta, x_{n-1} \sin \theta).
\]

Then \( \xi_n \) is a well-defined function since \( \rho_\theta \psi_{n-1} \rho_\theta^{-1}|_{B^{n-2}} = 1|_{B^{n-2}} \) for any \( \theta \). Let \( x \in B^n - B^{n-2} \). Then a sequence

\[
\left\{x^i = (x_1^i, \ldots, x_{n-2}^i, x_{n-1}^i \cos \theta^i, x_{n-1}^i \sin \theta^i)\right\}_{i=1}^\infty
\]

converges to
\[ x = (x_1, \ldots, x_{n-2}, x_{n-1} \cos \theta, x_{n-1} \sin \theta) \]

if and only if \( ((x'_1, \ldots, x'_{n-2}, x'_{n-1}))_{i=1}^{\infty} \) converges to \( (x_1, \ldots, x_{n-2}, x_{n-1}) \) and \( (\theta)_{i=1}^{\infty} \) converges to \( \theta \) up to modulo \( 2\pi \). Thus, the continuity of \( \zeta_n \) at \( x \in B^n - B^{n-2} \) is clear. Suppose \( x \in B^{n-2} \). Then a sequence \( (x'_i)_{i=1}^{\infty} \) converges to \( x \) if and only if \( (\rho_\theta^{-1}(x'_i))_{i=1}^{\infty} \) converges to \( x \), since \( d(x, \rho_\theta^{-1}(x'_i)) = d(x, x'_i) \) for each \( i \). Therefore \( \zeta_n \) is continuous at \( x \). Since the map \( \zeta'_n: S^n \to B^n \) defined by \( \zeta'_n|_{B^{n-1}} = \rho_\theta \psi_{n-1} \rho_\theta^{-1} \) is the inverse of \( \zeta_n \), \( \zeta_n \) is a homeomorphism. Furthermore, \( \zeta'_n|_{B^{n-1}} \) is the identity on \( \partial B^{n-1} \) for each \( \theta \) so that \( \zeta_n|_{\partial B^n} = 1_{\partial B^n} \) since \( \partial B^n \subset \bigcup_{0<\theta<\pi} \partial B^\theta \). \( \zeta_n \) is NEH since \( \zeta'_n|_{B^{n-1}} \) is NEH for each \( \theta \).

**Theorem 5.** A compact polyhedron \( P \) admits a NEH if and only if \( P \) contains no principal 1-cells.

**Proof.** To prove the necessity, suppose that \( P \) contains a principal 1-cell and suppose that there is a NEH \( h \) on \( P \). Let \( K \) be a triangulation of \( P \), \( K_1 \) the collection of principal 1-cells in \( K \) and write \( |K_1| = P_1 \). Then \( h(P_1) = P_1 \). Since \( P_1 \cap |K - K_1| \) is finite, the regular set of \( h|_{P_1} \) is at most finite. But by using the fact that the irregular set of a homeomorphism on either a circle or a closed 1-cell is nowhere dense, we can show that the irregular set of a homeomorphism on \( P_1 \) is nowhere dense. Therefore \( h \) cannot be a NEH on \( P \).

If \( P \) does not contain any principal 1-cell, then we can write \( P = \bigcup \{\sigma_j\}_{j=1}^k \) where \( \sigma_j \) is a principal \( n \)-simplex with \( n > 2 \) in some triangulation \( \{\sigma_i\}_{i=1}^m \). Therefore, since \( \zeta_n|_{\partial B^n} = 1_{\partial B^n} \), we can define a NEH \( h \) on \( P \) by taking \( h \) to be \( g_n \zeta_n g_n^{-1} \) on each principal cell \( \sigma_j \) of dimension \( n \) where \( g_n: B^n \to \sigma_j \) is a homeomorphism of \( B^n \) onto \( \sigma_j \).

**Lemma 6.** Let \( C \) be a locally connected contractible continuum in \( \text{Int}(B^2) \), where \( B^2 \subset \mathbb{R}^2 \). If \( C \) is nowhere dense in \( \mathbb{R}^2 \), then there is a map \( f \) from \( S^1 \) onto \( C \) such that the pair \( (M_f, C) \) is homeomorphic to \( (B^2, C) \) where \( M_f \) denotes the mapping cylinder associated with \( f \).

**Proof.** Since \( C \) is strongly cellular [5], there is a circle \( S \) and a homotopy \( H \) of \( S \) in \( \mathbb{R}^2 \) such that

1. \( H_0 \) is the identity,
2. \( H_t \) is an embedding for \( t < 1 \),
3. \( H_t(S) \cap H_u(S) = \emptyset \) for \( t \neq u \), and
4. \( H_1(S) = C \) [3, Theorem 2.1].

By the Schoenflies theorem, \( S \) bounds a disk. Therefore we may assume that \( S = S^1 \). It is clear, from the properties of \( H \), that \( H|_{S^1 \times [0,1]} \) is an embedding and \( \text{Im}(H) \subset B^2 \). To prove that \( \text{Im}(H) = B^2 \), suppose that there is \( x \in \text{Int}(B^2) - C \) such that \( \text{Im}(H) \subset B^2 - \{x\} \). Then there is a retraction \( \gamma: B^2 - \{x\} \to S^1 \). Now, \( \gamma H_t \) is homotopic to \( \gamma H_0 = 1_{S^1} \). But, since \( C \) is contractible, \( \gamma H_t \) is null homotopic. Thus, we obtain a contradiction. By taking \( f = H_1 \), we see that \( (M_f, C) \) is homeomorphic to \( (B^2, C) \).
Theorem 7. For each locally connected contractible continuum C which is nowhere dense in \( \text{Int}(B^2) \) with \( \text{diam}(C) > 0 \), there is a NEH \( h_C \) on \( B^2 \) such that \( \text{Fix}(h_C) = \{ x \in B^2 | h_C(x) = x \} = C \).

Proof. Let \( f \) be the map in Lemma 6. Since \( f \) is a closed map from \( S \) onto \( C \), \( C \) has the identification topology with respect to \( f \). Thus, \( (M_f, C) \) is homeomorphic to \( (S^1 \times I/\sim, \{ [e^{i\theta}, 1] | e^{i\theta} \in S^1 \}) \) where \( \sim \) is the equivalence relation on \( S^1 \times I \) induced by the map \( H \) which is defined in Lemma 6 and \( [x, t] \) denotes the equivalence class of \( (x, t) \). Write \( \{ [e^{i\theta}, 1] | e^{i\theta} \in S^1 \} = C' \). Then it suffices to show the existence of NEH \( h^* \) on \( S^1 \times I/\sim \) with \( \text{Fix}(h^*) = C' \). Let \( p: S^1 \times I \to S^1 \times I/\sim \) be the projection and \( h: S^1 \times I \to S^1 \times I \) be defined by \( h(e^{i\theta}, t) = (e^{i(\theta + \pi(1-t))}, t) \). Then, by the argument used in Lemma 1, \( h \) is a NEH on \( S^1 \times I \) and \( \text{Fix}(h) = S^1 \times \{ 1 \} \). Define \( h^*: S^1 \times I/\sim \to S^1 \times I/\sim \) by \( h^*([e^{i\theta}, t]) = ph(e^{i\theta}, t) \). Then, since \( h^* \) is a well defined one-to-one correspondence, it is a homeomorphism. Since \( p|_{S^1 \times \{ 0, 1 \}} \) is a homeomorphism and \( S^1 \times [0, 1] \) is compact for each \( t < 1 \), it is clear that \( \{ [e^{i\theta}, s] \in S^1 \times I/\sim | s < 1 \} \subset \text{Irr}(h^*) \). To show that \( [e^{i\theta}, 1] \in \text{Irr}(h^*) \), note first that \( \text{Fix}(h^*) = C' \) and \( \text{diam}(C') > 0 \). For each neighborhood \( U \) of \( [e^{i\theta}, 1] \), \( U \) contains \( [e^{i\theta}, t] \) for some irrational \( t \). Since the orbit of \( (e^{i\theta}, t) \) under \( h \) is dense in \( S^1 \times \{ t \} \), the orbit of \( [e^{i\theta}, t] \) under \( h^* \) is dense in \( \{ [e^{i\theta}, t] | 0 \leq \theta \leq 2\pi \} \). Now, if we take \( \delta = \frac{1}{2} \text{diam}(C') \), then we can find \( n \in Z \) such that \( d(h^*(n[e^{i\theta}, 1], h^*(e^{i\theta}, t)) > \delta \).

If \( h_1 \) and \( h_2 \) are topologically equivalent homeomorphisms then \( \text{Fix}(h_1) \) is homeomorphic to \( \text{Fix}(h_2) \). Consequently Theorem 7 implies the existence of uncountably many conjugacy classes of nowhere equicontinuous homeomorphisms.

References


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