MOVING HOLOMORPHIC DISKS OFF
ANALYTIC SUBSETS

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Abstract. Holomorphic maps of the unit disk into a complex manifold $X$, which miss an analytic subset $A$ of codimension $\geq 2$, are shown to be dense in all holomorphic maps of the disk into $X$. This implies that the Kobayashi pseudodistance on $X - A$ is the same as that on $X$, and thus leads to some new examples of nonhyperbolic manifolds containing no lines.

Let $X$ be a reduced complex space, let $A = \{z \in \mathbb{C}^l | |z| < 1\}$ be the unit disk, and denote by $\text{Hol}(A, A)$ the set of holomorphic maps of $A$ into $X$. If $A$ is a closed subset of $X$ and $\text{Hol}(A, X - A)$ is dense in $\text{Hol}(A, A)$ in the compact-open topology, then [2, p. 38] the restriction to $X - A$ of the Kobayashi pseudodistance on $X$ is the Kobayashi pseudodistance on $X - A$ (i.e., $d_X(p, q) = d_{X-A}(p, q)$ for $p, q \in X - A$, where $d_X$ and $d_{X-A}$ are the respective Kobayashi pseudodistances on $X$ and $X - A$). We prove

**Theorem 1.** If $X$ is a complex manifold and $A$ is (closed, and contained in) a closed analytic subset of $X$ of codimension $\geq 2$, then $\text{Hol}(A, X - A)$ is dense in $\text{Hol}(A, X)$ in the compact-open topology, and hence the Kobayashi pseudodistance on $X$ restricts to that on $X - A$.

Before the proof, several comments are in order. Theorem 1 is a generalization of results of [2], where the theorem is shown to hold if $X$ is a Stein manifold or, more generally, infinitesimally homogeneous. The proof we give uses this prior result and a result of Royden. Theorem 1 is best possible, in the sense that examples [2, pp. 37, 38] show that the conclusion is false if codimension $A = 1$ or if $X$ is singular. It is likely, however, that the hypotheses on $A$ can be weakened to: $A$ closed, and of first category in a nowhere-dense analytic subset of $X$. Furthermore, there is an obvious generalization: if $\text{codim} A \geq k + 1$, then $\text{Hol}(\Delta^k, X - A)$ is dense in $\text{Hol}(\Delta^k, X)$, where $\Delta^k$ is the unit $k$-dimensional polydisk. It can be proved by the same methods.

**Proof of the Theorem.** Obviously, we may assume that $A$ is a closed analytic subset of $X$ of codimension $\geq 2$. Let $f \in \text{Hol}(\Delta, X)$. We must show that $f$ can be approximated by maps of $\Delta$ into $X - A$. As a first step, let us show that we may assume that $f$ extends holomorphically past the boundary
of the unit disk. Define $f_t: \Delta \to X$ by $f_t(z) = f(tz)$, $0 < t < 1$. Then each $f_t$ extends past $\partial \Delta$ and $f_t \to f$ in the compact-open topology as $t \to 1$. Thus, if each $f_t$ is in the closure of $\text{Hol}(\Delta, X - A)$, so is $f$. Assuming, then, that $f$ has an extension $g: \Delta' \to X$, where $\Delta'$ is a disk of radius $> 1$, define $G: \Delta' \to \Delta' \times X$ by $G(z) = (z, g(z))$. Then $G$ is an embedding. Since $\Delta$ is a concentric subdisk of $\Delta'$ of strictly smaller radius, Lemma 3 of Royden’s paper [5] shows that there is a Stein open subset $U$ of $\Delta' \times X$, such that $G(\Delta) \subset U$. Let $A' = (\Delta' \times A) \cap U$. $A'$ is a closed analytic subset of $U$ of codimension $\geq 2$. Thus, the Stein manifold case of Theorem 1 shows that $\text{Hol}(\Delta, U - A')$ is dense in $\text{Hol}(\Delta, U)$. Let $\pi$ denote the restriction to $U$ of the projection of $\Delta' \times X$ onto $X$. Since the spaces involved are locally compact, composition with $\pi$ is a continuous map $\pi_*: \text{Hol}(\Delta, U) \to \text{Hol}(\Delta, X)$, which sends $\text{Hol}(\Delta, U - A')$ to $\text{Hol}(\Delta, X - A)$. Thus $f = \pi \circ (G|_\Delta)$ is in the closure of $\pi_*(\text{Hol}(\Delta, U - A')) \subset \text{Hol}(\Delta, X - A)$. □

As an application of Theorem 1 we will produce a new class of examples of nonhyperbolic manifolds containing no lines. Recall that a complex manifold $X$ is said to contain no lines if every holomorphic map $C \to X$ is constant. Since $d_C = 0$, and holomorphic maps do not increase pseudodistance, any hyperbolic manifold contains no lines. Brody [1] has shown that the converse is true for compact complex manifolds—that is, the hyperbolic ones are precisely the ones containing no lines. There are, however, noncompact, nonhyperbolic manifolds containing no lines; the first example was provided by D. Eisenman and L. Taylor [4, p. 130]. Now, let $F^d$ be the Fermat surface of degree $d$ in $\mathbb{P}^3$, given by the equation $z_0^d + z_1^d + z_2^d + z_3^d = 0$. $F^d$ is nonsingular. For any permutation $(i, j, k, l)$ of $(0, 1, 2, 3)$ and choices of $\mu$ and $\nu$, such that $\mu^d = -1 = \nu^d$, the line in $\mathbb{P}^3$ given by the equations $z_i = \mu z_j$, $z_k = \nu z_l$ lies in $F^d$. Each of these finitely many lines is biholomorphic to $\mathbb{P}^1$. By work of Mark Green [3, p. 70], for $d > 8$ any nonconstant map $C \to F^d$ has image lying in one of these lines. Let $d > 8$, and let $F^d$ be a surface obtained from $F^d$ by deleting finitely many points, in such a way that at least three points are removed from each of these lines. Since $\mathbb{P}^1$ with 3 or more points removed is hyperbolic, $F^d$ is a noncompact surface containing no lines. By Theorem 1, the Kobayashi pseudodistance between any two points remaining in one of these lines is still zero, so $F^d$ is not hyperbolic. Similar examples can be produced from any surface “containing only finitely many lines.”

REFERENCES


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