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MOVING HOLOMORPHIC DISKS OFF ANALYTIC SUBSETS

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ABSTRACT. Holomorphic maps of the unit disk into a complex manifold X, which miss an analytic subset A of codimension $\geqslant 2$, are shown to be dense in all holomorphic maps of the disk into X. This implies that the Kobayashi pseudodistance on $X - A$ is the same as that on X, and thus leads to some new examples of nonhyperbolic manifolds containing no lines.

Let X be a reduced complex space, let $\Delta = \{z \in \mathbf{C}^1 \mid |z| < 1\}$ be the unit disk, and denote by Hol (Δ, X) the set of holomorphic maps of Δ into X. If A is a closed subset of X and Hol $(\Delta, X - A)$ is dense in Hol (Δ, X) in the compact-open topology, then [2, p. 38] the restriction to $X - A$ of the Kobayashi pseudodistance on X is the Kobayashi pseudodistance on $X - A$ (i.e., $d_X(p,q) = d_{X-A}(p,q)$ for $p, q \in X - A$, where d_X and d_{X-A} are the respective Kobayashi pseudodistances on X and $X - A$). We prove

THEOREM 1. *If X is a complex manifold and A is* (*closed, and contained in*) *a closed analytic subset of X of codimension* $\geqslant 2$, *then* Hol $(\Delta, X - A)$ *is dense in* Hol (Δ, X) *in the compact-open topology, and hence the Kobayashi pseudodistance on X restricts to that on $X - A$.*

Before the proof, several comments are in order. Theorem 1 is a generalization of results of [2], where the theorem is shown to hold if X is a Stein manifold or, more generally, infinitesimally homogeneous. The proof we give uses this prior result and a result of Royden. Theorem 1 is best possible, in the sense that examples [2, pp. 37, 38] show that the conclusion is false if codimension $A = 1$ or if X is singular. It is likely, however, that the hypotheses on A can be weakened to: A closed, and of first category in a nowhere-dense analytic subset of X. Furthermore, there is an obvious generalization: if codim $A \geqslant k + 1$, then Hol $(\Delta^k, X - A)$ is dense in Hol (Δ^k, X), where Δ^k is the unit k-dimensional polydisk. It can be proved by the same methods.

PROOF OF THE THEOREM. Obviously, we may assume that A is a closed analytic subset of X of codimension $\geqslant 2$. Let $f \in$ Hol (Δ, X). We must show that f can be approximated by maps of Δ into $X - A$. As a first step, let us show that we may assume that f extends holomorphically past the boundary

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of the unit disk. Define $f_t\colon \Delta \to X$ by $f_t(z) = f(tz)$, $0 < t < 1$. Then each f_t extends past $\partial\Delta$ and $f_t \to f$ in the compact-open topology as $t \to 1$. Thus, if each f_t is in the closure of Hol $(\Delta, X - A)$, so is f. Assuming, then, that f has an extension $g\colon \Delta' \to X$, where Δ' is a disk of radius > 1, define $G\colon \Delta' \to \Delta' \times X$ by $G(z) = (z, g(z))$. Then G is an embedding. Since Δ is a concentric subdisk of Δ' of strictly smaller radius, Lemma 3 of Royden's paper [5] shows that there is a Stein open subset U of $\Delta' \times X$, such that $G(\Delta) \subset U$. Let $A' = (\Delta' \times A) \cap U$. A' is a closed analytic subset of U of codimension $\geqslant 2$. Thus, the Stein manifold case of Theorem 1 shows that Hol $(\Delta, U - A')$ is dense in Hol (Δ, U). Let π denote the restriction to U of the projection of $\Delta' \times X$ onto X. Since the spaces involved are locally compact, composition with π is a continuous map $\pi_*\colon$ Hol $(\Delta, U) \to$ Hol (Δ, X), which sends Hol $(\Delta, U - A')$ to Hol $(\Delta, X - A)$. Thus $f = \pi \circ (G|_\Delta)$ is in the closure of $\pi_*($Hol $(\Delta, U - A')) \subset$ Hol $(\Delta, X - A)$. □

As an application of Theorem 1 we will produce a new class of examples of nonhyperbolic manifolds containing no lines. Recall that a complex manifold X is said to contain no lines if every holomorphic map $\mathbf{C} \to X$ is constant. Since $d_{\mathbf{C}} \equiv 0$, and holomorphic maps do not increase pseudodistance, any hyperbolic manifold contains no lines. Brody [1] has shown that the converse is true for *compact* complex manifolds–that is, the hyperbolic ones are precisely the ones containing no lines. There are, however, noncompact, nonhyperbolic manifolds containing no lines; the first example was provided by D. Eisenman and L. Taylor [4, p. 130]. Now, let F^d be the Fermat surface of degree d in $\mathbf{P}^3$, given by the equation $z_0^d + z_1^d + z_2^d + z_3^d = 0$. F^d is nonsingular. For any permutation (i, j, k, l) of $(0, 1, 2, 3)$ and choices of μ and ν, such that $\mu^d = -1 = \nu^d$, the line in $\mathbf{P}^3$ given by the equations $z_i = \mu z_j$, $z_k = \nu z_l$ lies in F^d. Each of these finitely many lines is biholomorphic to $\mathbf{P}^1$. By work of Mark Green [3, p. 70], for $d > 8$ any nonconstant map $\mathbf{C} \to F^d$ has image lying in one of these lines. Let $d > 8$, and let $\tilde{F}^d$ be a surface obtained from F^d by deleting finitely many points, in such a way that at least three points are removed from each of these lines. Since $\mathbf{P}^1$ with 3 or more points removed is hyperbolic, $\tilde{F}^d$ is a noncompact surface containing no lines. By Theorem 1, the Kobayashi pseudodistance between any two points remaining in one of these lines is still zero, so $\tilde{F}^d$ is not hyperbolic. Similar examples can be produced from any surface "containing only finitely many lines."

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