ON $H$-CLOSED AND MINIMAL HAUSDORFF SPACES

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ABSTRACT. In this article, characterizations of $H$-closed and minimal Hausdorff spaces are given along with some relating properties.

1. Introduction. Letting $\mathcal{S}$ denote a class of topological spaces containing as a subclass the Hausdorff completely normal and fully normal spaces, Professors L. L. Herrington and P. E. Long, in a recent paper [2], gave the following characterization of $H$-closed spaces: A Hausdorff space $Y$ is $H$-closed if and only if for every space $X$ in class $\mathcal{S}$, each $g: X \rightarrow Y$ with a strongly-closed graph is weakly-continuous. In §3 of this paper we improve upon the sufficiency of this theorem by establishing that a Hausdorff space $Y$ is $H$-closed if for every space $X$ in class $\mathcal{S}$, each bijection $g: X \rightarrow Y$ with a strongly-closed graph is weakly-continuous.

Also, for a set $X$ and function $g: X \rightarrow X$, we let $F(g)$ denote the set of fixed points of $g$ (i.e. $F(g) = \{x \in X: x = g(x)\}$) and prove the following of our main theorems in §3.

(*) A Hausdorff space $(X, \tau)$ is $H$-closed if and only if for each topology $\tau^*$ on $X$ with $(X, \tau^*)$ in class $\mathcal{S}$ for which the identity function $i: (X, \tau^*) \rightarrow (X, \tau)$ has a strongly-closed graph, $F(g)$ is closed in $X$ for each bijection $g: (X, \tau^*) \rightarrow (X, \tau)$ with a strongly-closed graph.

(**) A Hausdorff space $(X, \tau)$ is $H$-closed if and only if for each topology $\tau^*$ on $X$ with $(X, \tau^*)$ in class $\mathcal{S}$ for which the identity function $i: (X, \tau^*) \rightarrow (X, \tau)$ has a strongly-closed graph, $F(g) = X$ whenever $F(g)$ is dense in $X$ and $g: (X, \tau^*) \rightarrow (X, \tau)$ has a strongly-closed graph.

In [3], Professors Herrington and Long have proved the following theorem: Let $g: X \rightarrow Y$ be a function and let $Y$ be minimal Hausdorff. If $g$ has a strongly-closed graph, then $g$ is continuous.

In §4 of this paper, we prove as another of our main results the following strong sufficiency to their theorem:

(***) A Hausdorff space $Y$ is minimal Hausdorff if for every space $X$ in class $\mathcal{S}$, each bijection $g: X \rightarrow Y$ with a strongly-closed graph is continuous.

In §5, we offer some examples.

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2. Preliminaries. We denote by $\text{cl}[K]$ the closure of a subset $K$ of a topological space.

2.1. Definition [6]. A point $x$ is in the $\theta$-closure of a subset $K$ of a space if each open subset $V$ of the space with $x \in V$ satisfies $K \cap \text{cl}[V] \neq \emptyset$. In this case we write $x \in \theta\text{-cl}[K]$.

2.2. Definition [6]. A point $x$ in a space is in the $\theta$-adherence of a filterbase $\mathcal{B}$ on the space if $x \in \theta\text{-cl}[F]$ for each $F \in \mathcal{B}$. In this case we will sometimes say that the filterbase $\mathcal{B}$ $\theta$-adheres to $x$ and use the notation $x \in \theta\text{-adh } \mathcal{B}$.

2.3. Definition [4]. A function $g: X \to Y$ is weakly continuous if for each $x \in X$ and each $F$ open in $Y$ about $g(x)$, there exists a $V$ open in $X$ about $x$ with $g(V) \subseteq \text{cl}[W]$.

We prove the following theorem which we use later in the paper.

2.1. Theorem. A function $g: X \to Y$ is weakly-continuous if and only if $g(\text{cl}[K]) \subseteq \theta\text{-cl}[g(K)]$ for each $K \subseteq X$.

Proof. Necessity. Let $y \in g(\text{cl}[K])$ where $K \subseteq X$ and $g: X \to Y$ is weakly-continuous. Let $x \in \text{cl}[K]$ with $g(x) = y$ and let $W$ be open about $y$. There is a $V$ open about $x$ satisfying $g(V) \subseteq \text{cl}[W]$. So

$$0 \neq g(V \cap K) \subseteq g(V) \cap g(K) \subseteq \text{cl}[W] \cap g(K)$$

and the necessity is proved.

Sufficiency. Suppose $g: X \to Y$ satisfies the inclusion of the theorem, let $x \in X$ and let $W$ be open in $Y$ about $g(x)$. Then $W \cap \theta\text{-cl}[g(X) - \text{cl}[W]] = \emptyset$. Consequently, $g(x) \notin \theta\text{-cl}[g(X - g^{-1}(\text{cl}[W]))]$. Thus

$$g(x) \notin g(\text{cl}[X - g^{-1}(\text{cl}[W])]) \quad \text{and} \quad x \notin \text{cl}[X - g^{-1}(\text{cl}[W])].$$

This implies that there is a $V$ open about $x$ satisfying $V \subseteq g^{-1}(\text{cl}[W])$ and the proof is complete.

2.4. Definition [2]. A function $g: X \to Y$ has a strongly-closed graph if for each $(x,y) \in G(g)$, the graph of $g$, there exist open sets $V \subseteq X$ and $W \subseteq Y$ containing $x$ and $y$, respectively, such that $(V \times \text{cl}[W]) \cap G(g) = \emptyset$.

We give without proof the following theorem which we use in the sequel.

2.2. Theorem. A function $g: X \to Y$ has a strongly-closed graph if and only if $\{g(x)\} = \cap_\Sigma \theta\text{-cl}[g(V)]$ for each $x \in X$ and each (some) open set base $\Sigma$ at $x$.

2.5. Definition. If $x_0$ is a point in a space $X$ and $\mathcal{B}$ is a filterbase on $X$, then $\{A \subseteq X: x_0 \in X - A \text{ or } F \cup \{x_0\} \subseteq A \text{ for some } F \in \mathcal{B}\}$ is a topology on $X$ which will be called the topology on $X$ associated with $x_0$ and $\mathcal{B}$. $X$ equipped with this topology will be called the space associated with $x_0$ and $\mathcal{B}$.

The space associated with a filterbase on a space and a point $x_0$ in the space will be used frequently in this paper. The following result is easily proved.

2.3. Theorem. Let $X$ be a space, let $x_0 \in X$ and let $\mathcal{B}$ be a filterbase on $X$ which has an empty intersection on $X - \{x_0\}$. The space $X$ associated with $x_0$ and $\mathcal{B}$ is in class $\mathcal{S}$.
3. H-closed spaces. We use the following characterization of H-closed spaces.

3.1. Definition [6]. A Hausdorff space is H-closed if each filterbase on the space $\theta$-adheres to some point in the space.

The sufficiency of our next theorem improves upon the sufficiency of the main result in [2]. We also give a different proof of the necessity of that main result based on the characterization of weakly-continuous functions in Theorem 2.1 above.

3.1. Theorem. A Hausdorff space $Y$ is H-closed if and only if for every space $X$ in class $\mathfrak{S}$, each bijection $g: X \rightarrow Y$ with a strongly-closed graph is weakly-continuous.

Proof. Strong necessity [2]. Let $X$ be any space, let $Y$ be H-closed, let $g: X \rightarrow Y$ have a strongly-closed graph and let $K \subset X$. For $y \in g(\text{cl } [K])$, choose $x \in \text{cl } [K]$ with $g(x) = y$ and let $\Sigma$ be an open set base at $x$. Then $\mathfrak{F} = \{g(V) \cap g(K): V \in \Sigma\}$ is a filterbase on $Y$. Consequently, $\theta$-adh $\mathfrak{F} \neq \emptyset$. Furthermore, $\theta$-adh $\mathfrak{F} \subset \{g(x)\} \cap \theta$-cl $[g(K)]$ by the properties of $\theta$-closure and Theorem 2.2 above (since $g$ has a strongly-closed graph).

Sufficiency. Let $Y$ be Hausdorff, let $x_0 \in Y$ and suppose $\mathfrak{F}$ is a filterbase on $Y$ which does not $\theta$-adhere to any point in $Y - \{x_0\}$. Let $X = Y$ be the space associated with $x_0$ and $\mathfrak{F}$. $X$ is in class $\mathfrak{S}$ by Theorem 2.3. Let $i: X \rightarrow Y$ be the identity function. If $x \neq y$ and $x \neq x_0$, choose $W$ open in $Y$ about $y$ with $x \notin \text{cl } [W]$. Then $\{x\}$ is open in $X$ and $((x) \times \text{cl } [W]) \cap G(i) = \emptyset$. If $x \neq y$ and $x = x_0$ then $y \neq x_0$, so there is an $A \in \mathfrak{F}$ and $W$ open about $y$ satisfying $x_0 \notin \text{cl } [W]$ and $F \cap \text{cl } [W] = \emptyset$. $F \cup \{x_0\}$ is open in $X$ and $((F \cup \{x_0\}) \times \text{cl } [W]) \cap G(i) = \emptyset$. We have proved that $i$ has a strongly-closed graph. Thus, $i$ is weakly-continuous at $x_0$ and by Theorem 2.1 we conclude that $i(\text{cl } [F]) \subset \theta$-cl $[F]$ for each $F \in \mathfrak{F}$. Since $x_0 \in \text{cl } [F]$ for each $F \in \mathfrak{F}$, the proof is complete.

We move now to two of our main results.

3.2. Theorem. A Hausdorff space $(X, \tau)$ is H-closed if and only if for each topology $\tau^*$ on $X$ with $(X, \tau^*)$ in class $\mathfrak{S}$ for which the identity function $i: (X, \tau^*) \rightarrow (X, \tau)$ has a strongly-closed graph, $F(g)$ is closed in $X$ for each bijection $g: (X, \tau^*) \rightarrow (X, \tau)$ with a strongly-closed graph.

Proof. Strong necessity. Let $(X, \tau)$ be H-closed and let $\tau^*$ be any topology on $X$ for which $i: (X, \tau^*) \rightarrow (X, \tau)$ has a strongly-closed graph. Let $g: (X, \tau^*) \rightarrow (X, \tau)$ be any function with a strongly-closed graph and let $v \in \text{cl } [F(g)]$; $g$ is weakly-continuous from Theorem 3.1. If $g(v) \neq v$, there are open sets $V \in \tau^*$ and $W \in \tau$ with $(v, g(v)) \in V \times W$ and $((V \times \text{cl } [W]) \cap G(i) = \emptyset$. This derives from the fact that $i$ has a strongly-closed graph. Since $g$ is weakly-continuous, there is an $A \in \tau^*$ with $v \in A$ and $g(A) \subset \text{cl } [W]$. $V \cap A \in \tau^*$ and $v \in V \cap A; g(V \cap A) \subset \text{cl } [W]$, so there is no $x \in V \cap A$ satisfying $g(x) = x$. This contradiction establishes the necessity.
Sufficiency. Suppose \( \mathcal{W} \) is a filterbase on \((X, \tau)\) which does not \( \theta \)-adhere to any point in \(X\). Choose \( x_0 \in X \) and let \( \tau^* \) be the topology on \(X\) associated with \( x_0 \) and \( \mathcal{W} \). Using the same proof as that of the sufficiency of Theorem 3.1, \((X, \tau^*)\) is in class \( \mathcal{S} \) and the identity function \( i: (X, \tau^*) \to (X, \tau) \) has a strongly-closed graph. Choose \( y_0 \in X - \{x_0\} \) and define \( g: (X, \tau^*) \to (X, \tau) \) by \( g(x_0) = y_0 \), \( g(y_0) = x_0 \) and \( g(x) = x \) otherwise; \( g \) is a bijection and we show that \( g \) has a strongly-closed graph. Let \((x, y) \in (X \times Y) - G(g)\). If \( x \neq x_0 \), choose \( W \in \tau \) with \( y \in W \) and \( g(x) \in \text{cl} \{W\} \). Then \((\{x\} \times \text{cl} \{W\}) \cap G(g) = \emptyset\). If \( x = x_0, y \neq y_0 \); so we may choose an \( F \in \mathcal{W} \) and a \( W \) open about \( y \) satisfying

\[
\{x_0\} \cup (\text{cl} \{W\} \cap (F \cup \{x_0, y_0\})) = \{x_0\};
\]

\[
((F \cup \{x_0\}) \times \text{cl} \{W\}) \cap G(g) = \emptyset.
\]

This completes the demonstration that \( g \) has a strongly-closed graph. We see easily that \( F(G) = X - \{x_0, y_0\} \) which is not \( \tau^* \)-closed. This contradiction completes the proof.

3.3. Theorem. A Hausdorff space \((X, \tau)\) is \( H \)-closed if and only if for each topology \( \tau^* \) on \(X\) with \((X, \tau^*)\) in class \( \mathcal{S} \) for which the identity function \( i: (X, \tau^*) \to (X, \tau) \) has a strongly-closed graph, \( F(g) = X \) whenever \( F(g) \) is dense in \(X\) and the function \( g: (X, \tau^*) \to (X, \tau) \) has a strongly-closed graph.

Proof. Strong necessity. In Theorem 3.2 we have found that for any topology \( \tau^* \) on \(X\) for which the identity function \( i: (X, \tau^*) \to (X, \tau) \) has a strongly-closed graph, \( F(g) \) is closed for any function \( g: (X, \tau^*) \to (X, \tau) \) with a strongly-closed graph. So, if \( F(g) \) is dense in \((X, \tau^*)\), we have \( F(g) = X \).

Sufficiency. We follow the proof of the sufficiency of Theorem 3.2 to the point immediately preceding the definition of \( g \). Choose \( y_0 \in X - \{x_0\} \) and define \( g: (X, \tau^*) \to (X, \tau) \) by \( g(x) = x \) if \( x \neq x_0 \), and \( g(x_0) = y_0 \). Using an argument similar to that in the proof of the sufficiency of Theorem 3.2 we can see that \( g \) has a strongly-closed graph. Then \( F(g) = X - \{x_0\} \) is dense in \(X\), a contradiction which completes the proof.

4. Minimal Hausdorff spaces. See [1] for a survey of minimal topological spaces. In this paper we use the following characterization of minimal Hausdorff spaces as a primitive.

4.1. Definition [3]. A Hausdorff space is minimal Hausdorff if each filterbase on the space with at most one \( \theta \)-adherent point is convergent.

Theorem 7 of [3] provides that a function into a minimal Hausdorff space must be continuous if the function has a strongly-closed graph. In [5], it is proved that a weakly-continuous function into a Hausdorff space has a closed graph. The following easily established theorem is analogous to the result in [5] and enables us to see that if a space is minimal Hausdorff the class of continuous functions into the space coincides with the class of functions into the space with strongly-closed graphs.
4.1. Theorem. If \( Y \) is Hausdorff and \( g: X \to Y \) is continuous, then \( g \) has a strongly-closed graph.

4.2. Theorem. Let \( Y \) be minimal Hausdorff. Then \( g: X \to Y \) is continuous if and only if \( g \) has a strongly-closed graph.

In our last theorem and the final of our main results, we give a strong sufficiency to Theorem 7 of [3]; we also give a different proof of Theorem 7 than that in [3].

4.3. Theorem. A Hausdorff space \( Y \) is minimal Hausdorff if and only if for each space \( X \) in class \( \mathcal{S} \), each bijection \( g: X \to Y \) with a strongly-closed graph is continuous.

Proof. Strong necessity [3]. Let \( Y \) be minimal Hausdorff, let \( X \) be any space, let \( g: X \to Y \) be any function with a strongly-closed graph and let \( K \subset X \). Let \( y \in g(\text{cl} \ K) \); choose \( x \in \text{cl} \ K \) with \( g(x) = y \) and let \( \Sigma \) be an open set base at \( x \). Then

\[
\bigcap_{\theta \in \Sigma} g(\text{cl} \ K) = \{ g(x) \}
\]

since \( \Theta = \{ g(V) \cap g(K) : V \in \Sigma \} \) is a filterbase on \( Y \), \( g \) has a strongly-closed graph and \( Y \) is \( H \)-closed. Since \( Y \) is minimal Hausdorff, we have \( \Theta \to y \). Thus, for any \( W \) open in \( Y \) about \( y \), there is a \( V \in \Sigma \) satisfying \( g(V) \cap g(K) \subset W \). Consequently, \( W \cap g(K) \neq \emptyset \) and \( y \in \text{cl} [g(K)] \).

Sufficiency. Let \( \Theta \) be a filterbase on \( Y \) with at most one \( \theta \)-adherent point \( x_0 \). Let \( X = Y \) be the space associated with \( x_0 \) and \( \Theta \), and let \( i: X \to Y \) be the identity function. By means of the argument used in the proof of the sufficiency of Theorem 3.1, we see that \( i \) has a strongly-closed graph. Thus \( i \) is continuous and if \( W \) is open in \( Y \) about \( x_0 \), there is an \( F \in \Theta \) with \( F \subset W \). Therefore, \( \Theta \to x_0 \) and the proof is complete.

5. Some examples. In this section, we give some examples to indicate some limitations on the weakening of hypotheses in the theorems in this paper. By way of notation, we let \( N \) denote the set of positive integers. For each \( k \in N \), we let \( Z(k) = \{ n \in N : n \geq k \} \), \( E(k) = \{ k + 1/2n : n \in N \} \), and \( O(k) = \{ k + 1/(2n - 1) : n \in N \} \).

5.1. Example. The hypothesis cannot be weakened to "closed graph" in either Theorem 3.1 or Theorem 4.3. Let \( Y = \{-1,0\} \cup \bigcup_{k=1}^{\infty} E(k) \cup \bigcup_{k=1}^{\infty} O(k) \cup N \)

with the topology generated by the following collection of sets as base: the subspace topology induced on \( \bigcup_{k=1}^{\infty} E(k) \cup \bigcup_{k=1}^{\infty} O(k) \cup N \) by the usual topology of the reals along with the collection of all sets of the form \( \{0\} \cup \bigcup_{k \geq m} E(k) \) and \( \{-1\} \cup \bigcup_{k > m} O(k) \), where \( m \in N \). Let \( X = Y \) with the topology associated with 1 and the filterbase \( \Theta = \{ Z(k) : k \in N \} \). \( Y \) is
minimal Hausdorff and $X$ is in class $\mathcal{S}$. The identity function $i: X \rightarrow Y$ has a closed graph but is not weakly-continuous at $x = 1$. $G(i)$ is not strongly-closed since $(V \times \overline{\text{cl}} \{W\}) \cap G(i) \neq \emptyset$ for any $V$ open about $1$ and $W$ open about $0$.

5.2. Example. The hypothesis cannot be weakened to “the identity function $i: (X, \tau^*) \rightarrow (X, \tau)$ has a closed graph” in either Theorem 3.2 or Theorem 3.3. Let $Y = \{0\} \cup \bigcup_{k=1}^{\infty} E(k) \cup N$ with the subspace topology from $Y$ in Example 5.1. Let $X = Y$ be the space associated with $1$ and the filterbase $\mathcal{F}$ in Example 5.1. Then $X$ is in class $\mathcal{S}$, $Y$ is $H$-closed and the identity function $i: X \rightarrow Y$ has a closed graph. Let $g: X \rightarrow Y$ be defined by $g(1) = 0$, $g(0) = 1$ and $g(x) = x$ otherwise. Then $g$ is a bijection and has a strongly-closed graph. However $F(g) = X - \{0,1\}$ is not closed in $X$. Now, let $h: X \rightarrow Y$ be defined by $h(x) = x$ if $x \neq 1$, and $h(1) = 0$. Then $h$ has a strongly-closed graph and $F(h) = X - \{1\}$ which is dense in $X$.

5.3. Example. The hypothesis cannot be weakened to “$g: (X, \tau^*) \rightarrow (X, \tau)$ with a closed graph” in either Theorem 3.2 or Theorem 3.3. Let $Y$ be the space in Example 5.2 and let $X = Y$ be the space associated with $0$ and the filterbase $\mathcal{F}$ from Example 5.1. The identity function $i: X \rightarrow Y$ has a strongly-closed graph. Let $g: X \rightarrow Y$ be defined by $g(0) = 1$, $g(1) = 0$ and $g(x) = x$ otherwise. Define $h: X \rightarrow Y$ by $h(x) = x$ if $x \neq 0$, and $h(0) = 1$. Then $g$ and $h$ have closed graphs which are not strongly-closed. $F(g) = X - \{0,1\}$ which is not closed in $X$ and $F(h) = X - \{0\}$ which is dense in $X$.

5.4. Example. “Weakly-continuous” cannot be replaced by “continuous” in Theorem 3.1. In Example 5.2, the function $g: X \rightarrow Y$ is a bijection with a strongly-closed graph; $g$ is not continuous at $x = 1$.

References


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